HIGHER STRUCTURE THEOREMS FOR CODIMENSION FOUR GORENSTEIN IDEALS.

JERZY WEYMAN

1. Introduction

The problem of classifying the Gorenstein ideals of codimension 4 was around ever since Buchsbaum-Eisenbud classification of Gorenstein ideals of codimension 3 was given in [10]. The first results were obtained by Kustin and Miller, who proved several important structure results, for example the existence of the associative multiplicative structure on the resolution of such ideal. There were several other approaches including analysis of linkage classes of Gorenstein ideals of codimension 4, the tight links. There were also results involving constructing classes of Gorenstein ideals and various constructions of Gorenstein ideals including the unprojections of Papadakis and Reid.

In this note we develop the theory of higher structure theorems for finite free resolutions of Gorenstein ideals of codimension 4 which brings out the spinor structure on the middle module of the resolution (see [48], [11]). This approach is analogous to an old idea of constructing "a generic ring" of finite free resolutions of a given format , see [6]. In the case of finite free resolutions the generic ring was constructed explicitly by Hochster [26] for resolutions of length 2. Similar approach was tried for resolutions of length 3 with partial success in [46], [56]. The approach was then completed in [58]. The new idea there was a realization that the defect Lie algebra conctructed in [56] was a positive part of the Kac-Moody Lie algebra associated to T-shaped graphs $T_{p,q,r}$, in a grading related to certain simple root.

The idea of the present paper is to apply similar methods to study the resolutions of Gorenstein ideals of codimension 4.

There is, however a marked difference between perfect codimension 3 and Gorenstein codimension 4 cases. The starting point of the procedure in codimension 3 case was the Buchsbaum-Eisenbud multiplier ring. However similar procedure in Gorenstein codimension 4 case could not work because it would never capture a spinor structure on the resolution of Gorenstein ideal of codimension 4 (see [11]). Thus in our approach we build in this structure from the beginning into our starting ring $A(n)_1$ which plays the role of the ring of Buchsbaum-Eisenbud multipliers.

We study the generic ring for the complex of length 3 we get from a self-dual complex of length 4 of format $(1, n, 2n-2, n, 1)$ by forgetting the right-most term. By proceeding in a complete analogy with [56], [58] we are able carry out a cycle killing process. There is a corresponding defect Lie algebra $\mathbb{L}_\bullet = \oplus_{I\geq 0}\mathbb{L}_i$ which is again a positive part of Kac-Moody Lie algebra associated this time to the graph E_n that is obtained from the graph D_{n-1} (corresponding to the spin group of a middle module in the resolution) by adding a node adjacent to node $n-1$. The main result of the paper is the construction of a sequence of factorizations p_i with defects $\mathbb{L}_i$. The first of these maps is related to the spinor structure, but this is, as it turns out, just the tip of the iceberg.

We apply this procedure to construct for each $n \geq 4$ the ring $A(n)_\infty$ which has a structure of a multiplicity free representation of the product of a Lie algebra $\underline{sl}_n \times \underline{g}(E_n)$.

The sequence of factorizations p_i is finite for $n \leq 8$, corresponding to the cases when the Kac-Moody Lie algebra of E_n is finite dimensional. This implies that the ring $A(n)_\infty$ is Noetherian for $n \leq 8$. Because of this we expect that the classification of Gorenstein ideals of codimension 4 will be much easier for $n \leq 8$. This new insight is also supported by some observations regarding the spinor coordinates.

In the second part of the note we start what we call "the E_n program" which would be completing the classification of Gorenstein ideals of codimension 4 with up to 8 generators in a way analogous to [58].

For $4 \leq n \leq 8$ we use the highest-lowest weight duality for the Lie algebra $\underline{g}(E_n)$ we construct a series of examples of Gorenstein ideals of codimension 4 which we conjecture will be generic, i.e. every other Gorenstein ideal of codimension 4 with n generators is a specialization of the generic one. The examples are the defining ideals of the affine pieces of certain Schubert varieties in the homogeneous spaces $G(E_n)/P_1$ where P_1 is a maximal parabolic corresponding to the vertex 1 in E_8 (in a Bourbaki notation).

The analogy with resolutions of length 3 goes further. In the spectrum of our generic ring $A(n)_\infty$ we can study the open set U_{Gor} where the full complex of length 4 is acyclic. This is analogous to the ideas of [59] where similar open set of points where the open set of points where the complex resolved a Cohen-Macaulay module was studied. The open set U_{Gor} in Noetherian cases has similar description to the set U_{CM} in a generic ring of resolutions of length 3, in terms of splitting of certain associated complex $\mathbb{F}_\bullet^{top}$. The ideals that are specializations of the defining ideals of Schubert varieties in homogeneous spaces can be defined intrinsically in terms of the resolution. This pattern might even go beyond the Noetherian cases.

Throughout we work over an algebraically closed field K of characteristic zero in order to be able to use representation theory. But the Schubert varieties are characteristic free so they should be generic examples in a characteristic free way.

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2. Representation theory background

2.1. **General linear group.** Let V be a vector space over a field K (or a free module over a ring R). We will use the following notation for the representations of the group $GL_n(V)$. For the dominant integral weight $(a_1, \cdots, a_n)$ where $a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \geq \cdots \geq a_n$, $S_{(a_1,\cdots,a_n)}V$ denotes the corresponding Schur module. The main reference we will be using is the book [19], lecture 6.

2.2. **Preliminaries on representations of the spin groups.** We first describe the results over an algebraically closed field of characteristic zero. Then we indicate which of them stay true over a filed of characteristic different than 2.

2.2.1. *Characteristic zero case.* We work over an algebraically closed field K of characteristic zero.

Most of the material in this section can be found in the book of Fulton and Harris [19], in lectures 18-20. Other references are [22, Chapters 2,3,6], [32, Section 2.15], and [15, Chapter 2]. Let V be an orthogonal space of dimensions $2m$ over K. We put the quadratic form $Q : V \otimes V \to K$ in the hyperbolic form. More precisely, let W be an isotropic space in V of dimension m. We can identify V with $W \oplus W^*$ and the quadratic form Q with the duality

$$Q : W \otimes W^* \to K,$$

also requiring W and W^* being isotropic.

Sometimes we write $\langle v, w\rangle$ instead of $Q(v, w)$ for $v, w \in V$. Throughout we deal with the representations of the special orthogonal Lie algebra $\underline{so}(V)$, as it is well known that the categories of rational representations of the spin group $Spin(2m)$ and of $\underline{so}(V)$ are equivalent. The maximal toral subalgebra in the Lie algebra $\underline{so}(V)$ is the maximal toral subalgebra of diagonal matrices in $\underline{gl}(W)$. It consists of matrices

$$\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

where A is an $m \times m$ diagonal matrix. We denote the basis of V as follows. Vectors $\{e_1, \dots, e_m\}$ are a basis of W, and $\{e_{-1} = e_1^*, \dots, e_{-m} = e_m^*\}$ form the dual basis in W^*. Their weights are respectively ϵ_i and $-\epsilon_i$ for $1 \le i \le m$.

Since the quadratic form Q is in standard form in any characteristics different from 2, we can use representation of $\mathrm{Spin}(V)$ as well. For the representation of the spin group, we use the following notation. In this case, a maximal torus H of $\mathrm{Spin}(V)$ and $\mathfrak{h} = \mathrm{Lie}(H)$ are

$$H = \{\mathrm{diag}[x_1, \cdots, x_m, x_m^{-1}, \cdots, x_1^{-1}] : x_i \in K \setminus \{0\}\},$$

$$\mathfrak{h} = \{\mathrm{diag}[a_1, \cdots, a_m, -a_m, \cdots, -a_1] : a_i \in K\}.$$

For $i = 1, \dots, m$, define $\langle \varepsilon_i, D\rangle = a_i$ where $D = \mathrm{diag}[a_1, \cdots, a_m, -a_m, \cdots, -a_1]$ is in $\mathfrak{h}$. Then $\{\varepsilon_1, \dots, \varepsilon_m\}$ is a basis for $\mathfrak{h}^*$. All representations of $Spin(V)$ restrict to H so they decompose to weights with respect to H.

Let $e_{i,j}$ be the matrix that takes e_j to e_i and annihilates e_k for $k \neq j$ where $i, j \in \{\pm 1, \cdots, \pm m\}$. Set $X_{\varepsilon_i - \varepsilon_j} = e_{i,j} - e_{-j,-i}$ and $X_{\varepsilon_i + \varepsilon_j} = e_{i,-j} + e_{j,-i}$ for $1 \le i, j \le m$, for $i \neq j$. Then $[D, X_{\pm(\varepsilon_i - \varepsilon_j)}] = \pm\langle \varepsilon_i - \varepsilon_j, D\rangle X_{\pm(\varepsilon_i - \varepsilon_j)}$ and $[D, X_{\pm(\varepsilon_i + \varepsilon_j)}] = \pm\langle \varepsilon_i + \varepsilon_j, D\rangle X_{\pm(\varepsilon_i + \varepsilon_j)}$. Thus $\pm(\varepsilon_i - \varepsilon_j)$ and $\pm(\varepsilon_i + \varepsilon_j)$ for $1 \le i < j \le m$ are the roots, and the associated set of positive roots are $\{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \le i < j \le m\}$.

Let us recall that when K is algebraically closed of characteristic zero, then irreducible representations of $\underline{so}(V)$ are parametrized by dominant integral weights

$$\lambda = \sum_{i-1}^{m} \lambda_i \omega_i$$

where $\lambda_i \in \mathbf{Z}_{\geq 0}$ and ω_i are so-called fundamental weights. We denote $V(\lambda)$ the irreducible representation corresponding to highest weight λ.

The fundamental weights of the orthogonal Lie algebra $\mathfrak{so}(V, \langle,\rangle)$ are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \le i \le m-2$, and $\omega_{m-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} + \varepsilon_m)$, $\omega_m = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} - \varepsilon_m)$.

The fundamental representations of $Spin(2m)$ are as follows. For $1 \leq i \leq m-2$ we have

$$V(\omega_i) = \bigwedge^i V.$$

For $i = m-1, m$ the fundamental representations are the half-spinor representations. To define them we need a Clifford algebra

$$C(V,Q) = T(V)/I(V)$$

where $T(V)$ is a tensor algebra of V and $I(V)$ is the two-sided ideal in $T(V)$ generated by the elements

$$v_1 \otimes v_2 + v_2 \otimes v_1 - 2Q(v_1, v_2)$$

for $v_1, v_2 \in V$. Note that since the ideal $I(V)$ has generators with components in the 0-th and 2-nd graded component of $T(V)$, the Clifford algebra decomposes to its even part $C(V)_+$ and its odd part $C(V)_-$. Additively we have decompositions

$$C(V)_+ = \oplus_{i \text{ even}} \bigwedge^i V, C(V)_- = \oplus_{i \text{ odd}} \bigwedge^i V.$$

Let $f = e \wedge \ldots \wedge e_m$.

We have (see [19], lecture 20 for more details, note that our convention interchanges W and W^*).

Proposition 2.1. *The left ideal*

$$S = C(Q)^{\cdot} f$$

is additively isomorphic to the exterior algebra $\bigwedge^\bullet W^*$. It is therefore a representation of $\mathfrak{so}(V)$. It decomposes to even and odd parts $S_+ := \bigwedge^{even} W^*$ and $S_- := \bigwedge^{odd} W^*$. S is called a Clifford module, and S_+ and S_- are called half-spinor modules.

We also have

$$V(\omega_{m-1}) = S_+, V(\omega_m) = S_-.$$

Both half-spinor representations have dimension 2^{m-1}. Let $L \subset [1,m]$ be a subset, let L' be its complement. We denote by u_L a coset of the tensor $\wedge_{i\in L} w_{-i}$. This is a weight vector of weight $-\frac{1}{2}(\sum_{i\in L} \epsilon_i + \sum_{i\in L'} \epsilon_i)$.

For the convenience of the reader we describe the action of the Lie algebra $\mathfrak{so}(V)$ on half-spinor representations. Strictly speaking it will not be needed but it explains weight decompositions of half-spinor representations.

For $a, b \in V$, define $R_{a,b} \in \mathrm{End}(V)$ as $R_{a,b}v = \langle b, v\rangle a - \langle a, v\rangle b$. By [15, Section 2.4], $R_{a,b}$ spans $\mathfrak{so}(V, \langle\ ,\ \rangle)$ for $a, b \in V$. Then $R_{e_i,e_j} = e_{-i,j} - e_{-j,i}$ where $e_{i,j}$ be an elementary transformation on V that carries e_i to e_j and others to 0.

For $y^* \in W^*$, the exterior product $\boldsymbol{\epsilon}(y^*)$ and the interior product operator $\mathfrak{i}(y)$ on $\bigwedge W$ are defined as $\boldsymbol{\epsilon}(y^*)x^* = y^* \wedge x^*$ and

$$\mathfrak{i}(y)(y_1^* \wedge \cdots \wedge y_k^*) = \sum_{j=1}^{k} (-1)^{j-1} \langle y, y_j^*\rangle y_1^* \wedge \cdots \wedge \widehat{y_j^*} \wedge \cdots \wedge y_k^*$$

where $y_i^* \in W^*$, $x^* \in \bigwedge W^*$ and $\widehat{y_j^*}$ means to omit y_j^*.

Define linear maps $\boldsymbol{\gamma} : V \to \mathrm{End}(\bigwedge W^*)$ as $\boldsymbol{\gamma}(y + y^*) = \mathfrak{i}(y) + \boldsymbol{\epsilon}(y^*)$ for $y \in W$ and $y^* \in W^*$, and $\boldsymbol{\varphi} : \mathfrak{so}(V, \langle\ ,\ \rangle) \to \mathrm{Cliff}_2(V, \langle\ ,\ \rangle)$ as $\boldsymbol{\varphi}(R_{a,b}) = \frac{1}{2}[\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)]$ for $a, b \in V$ where $[\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)] = \boldsymbol{\gamma}(a)\boldsymbol{\gamma}(b) - \boldsymbol{\gamma}(b)\boldsymbol{\gamma}(a)$. By [15, Chapter 2], $\boldsymbol{\varphi}$ is injective, and the Lie algebra of $\mathrm{Spin}(V)$ is $\boldsymbol{\varphi}(\mathfrak{so}(V, \langle\ ,\ \rangle))$.

Let us also look at other exterior powers of V. We have

$$\bigwedge^{m-1} V = V(\omega_{m-1} + \omega_m),$$

$$\bigwedge^{m} V = V(2\omega_{m-1}) \oplus V(2\omega_m).$$

To see the decomposition in the second formula, we proceed as follows. Let $\tilde{Q} : V \to V^*$ be an $\underline{so}(V)$-equivariant isomorphism defined by the formula

$$\tilde{Q}(v_1)(v_2) := Q(v_1, v_2).$$

This isomorphism defines a similar $\underline{so}(V)$-equivariant isomorphism

$$\bigwedge^m \tilde{Q} : \bigwedge^m V \to \bigwedge^m V^*.$$

We also have an $\underline{sl}(V)$-equivariant isomorphism

$$\phi : \bigwedge^m V^* \to \bigwedge^m V,$$

using $e_1^* \wedge \ldots \wedge e_m^* \wedge e_1 \wedge \ldots \wedge e_m$ as a volume form. We define an $\underline{so}(V)$-equivariant isomorphism

$$\tau = \phi \circ (\bigwedge^m \tilde{Q}) : \bigwedge^m V \to \bigwedge^m V.$$

One proves easily that $\tau^2 = 1$. The representation $V(2\omega_{m-1})$ can be identified with the 1-eigenspace of τ and $V(2\omega_m)$ can be identified with the -1-eigenspace of τ. Thus the operators $\frac{1}{2}(\tau - 1)$ and $\frac{1}{2}(\tau + 1)$ are the projections on both direct summands.

We will also need the morphisms

$$i_{m-1} : V(\omega_{m-1}) \to V \otimes V(\omega_m), p_{m-1} : V \otimes V(\omega_m) \to V(\omega_{m-1})$$

And analogous morphisms

$$i_m : V(\omega_m) \to V \otimes V(\omega_{m-1}), p_m : V \otimes V(\omega_{m-1}) \to V(\omega_m).$$

It is enough to define p_{m-1}, p_m as the inclusions are their duals (one has to be careful whether this duality does not switch them-this depends on parity of m). To define p_{m-1}, p_m we notice that, decomposing $V = W \oplus W^*$ the subspace W will act on $\bigwedge^\bullet W$ by exterior multiplication and the subspace W^* will act on $\bigwedge^\bullet W$ by contraction. More explicitly, we have

$$p_{m-1}(e_i \otimes t) = w_i \wedge t, p_{m-1}(e_i^* \otimes t) = w_i^*(t),$$

for $t \in \bigwedge^{odd} W$. Similarly,

$$p_m(e_i \otimes t) = w_i \wedge t, p_{m-1}(e_i^* \otimes t) = w_i^*(t),$$

for $t \in \bigwedge^{even} W$.

Let me also mention the tensor product decompositions that will be useful

Proposition 2.2. *(1)*

$$\bigwedge^2 V(\omega_1) = V(\omega_2)$$

$$S_2 V(\omega_1) = V(2\omega_1) \oplus K$$

$$\bigwedge^2 V(\omega_{m-1}) = \oplus_i V(\omega_{m-2-4i})$$

$$S_2 V(\omega_{m-1}) = V(2\omega_{m-1}) \oplus \oplus_i V(\omega_{m-4i})$$

$$\bigwedge^2 V(\omega_m) = \oplus_i V(\omega_{m-2-4i})$$

$$S_2 V(\omega_m) = V(2\omega_m) \oplus \oplus_i V(\omega_{m-4i})$$

with the convention that $V(\omega_0) = K$.

(2) We also have

$$V(\omega_1) \otimes V(\omega_{m-1}) = V(\omega_1 + \omega_{m-1}) \oplus V(\omega_m),$$

$$V(\omega_1) \otimes V(\omega_m) = V(\omega_1 + \omega_m) \oplus V(\omega_{m-1}),$$

$$V(\omega_{m-1}) \otimes V(\omega_m) = V(\omega_{m-1} + \omega_m) \oplus \oplus_{i \geq 1} V(\omega_{m-1-2i})$$

again with the convention that $V(\omega_0) = K$.

The important formula for us will be

We need a result (known as Klimyk's formula) on calculating tensor products of irreducible representations of reductive groups.

Proposition 2.3. *Let λ and μ be two dominant weights for a linearly reductive group G. Let us write the character of $V(\mu)$ as*

$$char(V(\mu)) = \sum_{\nu} mult(\nu, V(\mu)) e^{\nu}.$$

Then the highest weights (with multiplicities) occurring in $V(\lambda) \otimes V(\mu)$ can be calculated as follows. Take all the weights of the form $\lambda + \nu$ where ν is a weight in $V(\mu)$ (with multiplicities) and then apply Bott's algorithm to get $\pm\eta$ where η is a dominant weight or zero. Then sum up the obtained weights $\pm\eta$. We will get a nonnegative linear combination of dominant integral weights which will give us decomposition of $V(\lambda) \otimes V(\mu)$.

When working over a field K of characteristic $\neq 2$ we will use all representations $V(\omega_i)$ constructed above and the equivariant maps constructed directly. These representations might not be irreducible but this will not concern us.

2.2.2. *Finite characteristic different than* 2. Let us work over an algebraically closed field of characteristic different than 2. We work with the quadratic form Q which is already in hyperbolic form, so we can set again $V = W \oplus W^*$ and use the basis $\{e_1, \dots, e_m, e_{-1}, \dots, e_{-m}\}$. The representations $V(\omega_i)$ for $1 \le i \le m-2$ can be constructed over K. They might not be irreducible but this will not be relevant. The construction of half-spinor representations also can be carried out over K. Among the tensor product decompositions that were listed the most important are the ones for the symmetric powers of half-spinor representations. These formulas are related to the homogeneous spaces $Spin(2m)/P_{m-1}$ and $Spin(2m)/P_m$ which are two connected components of the isotropic Grassmannian $IGrass(m, V)$. These spaces are closed subvarieties of the projective spaces $\mathbf{P}(V(\omega_{m-1})$ and $\mathbf{P}(V(\omega_m))$ respectively. The Plücker embeddings are the doubles of these fundamental embeddings. The equations defining the subvariety $Spin(2m)/P_{m-1}$ inside of $\mathbf{P}(V(\omega_{m-1}))$ (resp. $Spin(2m)/P_m$ inside of $\mathbf{P}(V(\omega_m)))$ are quadratic. They are well known in commutative algebra. The big open cell inside of $Spin(2m)/P_{m-1}$ can be identified with the space of $m \times m$ skew-symmetric matrices and the restrictions of spinor coordinates are the sub-Pfaffians of all sizes of these skew-symmetric matrices. So the quadratic equations defining the subvariety $Spin(2m)/P_{m-1}$ inside of $\mathbf{P}(V(\omega_{m-1})$ are all quadratic equations on sub-Pfaffians of a generic skew-symmetric $m \times m$ matrix. This whole construction is characteristic free. In degree 2 we see that $V(2\omega_{m-1})$ is a factor of $S_2(V(\omega_{m-1}))$ by the span of these quadratic equations. This is an analogue of the decomposition of $S_2(V(\omega_{m-1}))$ in positive characteristic.

The decomposition of $\bigwedge^m V$ into two summands of equal dimension as described in previous section is also true over K. The summands might not be irreducible, but this will not be relevant.

2.2.3. *Certain Spin(V)-equivariant map* $\mathbf{p}$ *and its properties.* Let V be an orthogonal space of rank $2m$ over algebraically closed field K of characteristics different from 2. Our goal in this section is to describe certain equivariant map $\mathbf{p} : S_2(V(\omega_m)) \to \bigwedge^m V$ explicitly. If K has characteristic zero, then by formula 2.2, we have a unique such Spin(V)-equivariant map $\mathbf{p}$ up to scalar. Over fields K of characteristic different from 2 one can check that formulas we write down below define an equivariant map, so we will just use it.

Remark 2.4. *The map* $\mathbf{p}$ *will be very important in our application as it will give polynomial formula expressing arbitrary Buchsbaum-Eisenbud multipliers by quadratic expressions involving spinor coordinates.*

Before we start we need some notation. The signature of a permutation of the set $[1, m]$, denoted by sgn, is a multiplicative map from the group of permutations S_m to ± 1. Permutations with signature $+1$ are even and those with sign -1 are odd. Also L^c denotes the complement of a subset L of $[1, m]$.

Lemma 2.5. Set $q = \lfloor \frac{m}{2} \rfloor$. Let $J_{2k} = \{\gamma_1, \dots, \gamma_{2k}\}$ with $1 \le \gamma_1 < \dots < \gamma_{2k} \le m$, $1 \le k \le q$. Let $\mathbf{p} : S_2(V(\omega_m)) \to \bigwedge^m V$ be an equivariant map such that $\mathbf{p}(u_\phi u_\phi) = e_1 \wedge e_2 \wedge \dots \wedge e_m$. Then we have

$$\mathbf{p}(u_{J_{2k}} u_\phi) = \frac{1}{2^{\ell(J_{2k})-1}} \sum_{L \subset J_{2k}, \ell(J_{2k}) = 2\ell(L)} \mathrm{sgn}(J_{2k}, L) e_{-L} \wedge e_{J_{2k}^c} \wedge e_L \tag{1}$$

where J_{2k}^c is the complement of J_{2k} in $[1,m]$, $e_{-L} = \bigwedge_{i\in L} e_{-l}$, $u_{J_{2k}} = e_{-\gamma_1} \wedge e_{-\gamma_2} \wedge \cdots \wedge e_{-\gamma_{2k}}$, $1 \le k \le q$, $e_L = \bigwedge_{i\in L} e_l$, $\mathrm{sgn}(J_{2k}, L)$ is the signature of permutations of J_{2k}, and $\ell(J)$ is the length of any indexing set $J \subset [1,m]$.

Proof. We prove formula (1) by reverse induction on q. Then

$$\mathbf{p}(u_{J_{2q}} u_\phi) = \frac{1}{2^{\ell(J_{2q})-1}} \sum_{L\subset J_{2q}, \ell(J_{2q})=2\ell(L)} \mathrm{sgn}(J_{2q}, L) e_{-L} \wedge e_{J_{2q}^c} \wedge e_L$$

For $i < j$, we see that $\mathfrak{i}(w_{\gamma_i})\mathfrak{i}(w_{\gamma_j})(u_{J_{2q}} u_\phi) = (-1)^{r_{\gamma_i}+r_{\gamma_j}} u_{J_{2q}\setminus\{\gamma_i,\gamma_j\}} u_\phi$ since $\mathfrak{i}(w_{\gamma_i})\mathfrak{i}(w_{\gamma_j})$ acts on $V(2\omega_m)$. But action of $\mathfrak{i}(w_{\gamma_i})\mathfrak{i}(w_{\gamma_j})$ on $V(2\omega_m)$ corresponds to an action of R_{e_i,e_j} on $\bigwedge^m V$. Then $\mathbf{p}(u_{J_{2q}\setminus\{\gamma_i,\gamma_j\}} u_\phi)$ is of the form

$$\frac{1}{2^{2q-3}} \sum_{L\subset J_{2q}\setminus\{\gamma_i,\gamma_j\}, \ell(L)=q-1} \mathrm{sgn}(J_{2q} \setminus \{\gamma_i,\gamma_j\}, L) e_{-L} \wedge e_{(J_{2q}\setminus\{\gamma_i,\gamma_j\})^c} \wedge e_L$$

since the following diagram

$$\begin{array}{ccc} V(2\omega_m) & \xrightarrow{\mathbf{p}} & \bigwedge^m V \\ {\scriptstyle \mathrm{i}(\boldsymbol{e}_i)\mathrm{i}(\boldsymbol{e}_j)}\downarrow & & \downarrow{\scriptstyle R_{e_i,e_j}} \\ V(2\omega_m) & \xrightarrow{\mathbf{p}} & \bigwedge^m V \end{array}$$

commutes because the map $\mathbf{p}$ is equivariant, [22, Lemma 6.2.1]. Applying interior products successively, on gets expression for $k = 1$ as

$$\mathbf{p}(u_{\{\gamma_i,\gamma_j\}} u_\phi) = \frac{\mathrm{sgn}(\{\gamma_i,\gamma_j\}, \gamma_i, \gamma_j)}{2} (e_{-\gamma_i} \wedge e_{\{\gamma_i,\gamma_j\}^c} \wedge e_{\gamma_i} + e_{-\gamma_j} \wedge e_{\{\gamma_i,\gamma_j\}^c} \wedge e_{\gamma_j})$$

Again, by applying internal product, we obtain

$$\mathbf{p}(u_\phi u_\phi) = e_1 \wedge \cdots \wedge e_m.$$

By setting $\mathbf{p}|_{\bigwedge^i V} = 0$, we get $\mathbf{p} : S_2(V(\omega_m)) \to \bigwedge^m V$. $\square$

Remark 2.6. Let $L, M \subset [1,m]$ of even cardinality. Set $L \ominus M = (L \setminus M) \cup (M \setminus L)$. Assume that $L \ominus M$ is nonempty. Note that $L \ominus M$ is of even cardinality. Using Lemma 2.5, one can evaluate the map $\mathbf{p}$ by permuting indices of the monomial $u_L u_M$ as

$$\frac{1}{2^{\ell(L\ominus M)-1}} \sum_{J\subset L\ominus M, \ell(L\ominus M)=2\ell(J)} \mathrm{sgn}(L \cup M, J) e_{-(L\cap M)} \wedge e_{-J} \wedge e_{L^c\cap M^c} \wedge e_J.$$

Moreover, $\mathbf{p}(u_L u_L) = \mathrm{sgn}(L, L^c) e_{-L} \wedge e_{L^c}$.

3. Generalities and notation

Let R be a commutative ring (assumed graded or local) and let I be a Gorenstein ideal of codimension 4 (assumed homogeneous if R is graded). Then the minimal free resolution of R/I as an R-module has the form

$$\mathbb{F}_\bullet : 0 \to R \xrightarrow{d_1^*} F_1^* \xrightarrow{d_2^*} F_2^* \cong F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R$$

where F_1 is a free R-module of dimension n, F_2 is a free R-module of dimension $2n-2$, d_2, d_1 are R-linear maps, and the duality $F_2^* \cong F_2$ is given via the Gorenstein duality, i.e. there exists a symmetric nondegenerate form on F_2. This form is part of an associative multiplicative structure on $\mathbb{F}_\bullet$ which is associative, graded commutiative and satisfies Leibniz rule [39]. The resolution $\mathbb{F}_\bullet$ satisfies the first Buchsbaum-Eisenbud structure theorem so there exists a map $a_3 : R \to \bigwedge^{n-1} F_2$ such that we have a commutative diagram

$$\begin{tikzcd} \bigwedge^{n-1} F_1^* \arrow[rr, "\bigwedge^{n-1} d_3"] \arrow[dr, "d_4^*"'] & & \bigwedge^{n-1} F_2^* \\ & R \arrow[ur, "a_3"'] & \end{tikzcd}$$

We make an assumption that the nondegenerate symmetric bilinear map $F_2 \otimes F_2 \to R$ can be brought over R to the hyperbolic form. This requires the assumption that 2 is invertible in R and that R is either complete regular or a polynomial ring over a filed of characteristic $\neq 0$. In fact it is an open problem whether over a regular local ring any nondegenerate symmetric bilinear map can be brought to a hyperbolic form (see [1] and references therein).

Under this assumption we have a spinor structure (see [11]) on $\mathbb{F}_\bullet$. This means that the image of the map a_3 is actually in one of the subrepresentations $(\bigwedge^{n-1} F_2)^+$ or $(\bigwedge^{n-1} F_2)^-$ of $\bigwedge^{n-1} F_2$ (and we have a choice in which one, we choose $(\bigwedge^{n-1} F_2)^+$), and there exists a map $\tilde{a}_3 : R \to V(\omega_{n-2}, D_{n-1})$ such that the diagram

$$\begin{tikzcd} R \arrow[rr, "S_2\tilde{a}_3"] \arrow[dr, "a_3"'] & & S_2(V(\omega_{n-2}, D_{n-1})) \arrow[dl, "p"] \\ & (\bigwedge^{n-1} F_2)^+ & \end{tikzcd}$$

commutes. Here $V(\omega_{n-2}, D_{n-1})$ is a half-spinor representation for the root system D_{n-1}. and p is a natural projection.

4. The ring $A(n)_1$

We create the "generic form" of the situation described in the last section.

Let K be an algebraically closed field of characteristic zero. Let F be a vector space of dimension n over K and let G be an orthogonal space of dimension $2n-2$. This means there is a non-degenerate symmetric bilinear form $\langle , \rangle$ on G. We identify G^* and G via the symmetric form $\langle , \rangle$. We also denote by H a one dimensional vector space (used only to stress the functorial meaning of some terms).

For a $GL(F)$ highest weight $(a_1, \ldots, a_n)$ ($a_i \in \mathbf{Z}$, $A(n)_1 \geq a_2 \geq \ldots \geq a_n$) we will denote $S_{(a_1,\ldots,a_n)}F$ the highest weight irreducible representation of $GL(F)$. If $a_n \geq 0$ this is a Schur functor, otherwise it is a negative power of determinant representation tensored with a Schur functor. For the highest weight $\mu = (\mu_1, \ldots, \mu_{n-1})$ for the root system of type D_{n-1} (here μ_i are either integers or half-integers). We also denote $V_\mu(G)$ the irreducible highest weight representation for the spin group $Spin(G)$.

We will be dealing with the Grassmannian $Grass(1, F)$ and with isotropic Grassmannian $IGrass(n-1, G)$. We denote

$$0 \to \mathcal{R} \to F \times Grass(1, F) \to \mathcal{Q} \to 0$$

the tautological sequence on $Grass(1, F)$ and

$$0 \to \mathcal{S} \to G \times IGrass(n-1, G) \to \mathcal{S}^* \to 0$$

the tautological sequence on $IGrass(n-1, G)$.

We denote by X the affine space of pairs of linear maps $\phi : K \to F$ and $\psi : F \to G^* \cong G$, $a_3 : R \to (\bigwedge^{n-1} G)^+$. The coordinate ring $A = K[X]$ is canonically identified with $Sym(F^*) \otimes Sym(F \otimes G)$.

We define the subvariety

$$Y' = \{(\phi, \psi) \in X \mid \psi\phi = 0, rank\ \psi \le n-1, \langle , \rangle|_{Im(\psi)} = 0, \}.$$

The variety Y' has a natural desingularization

$$Z'' = \{(\phi, \psi, \mathcal{R}, \mathcal{S}) \mid Im(\phi) \subset \mathcal{R}, \subset Ker(\psi), Im(\psi) \subset \mathcal{S}\}$$

We have

$$\mathcal{O}_{Z''} = Sym(\mathcal{R}^* \otimes Sym(\mathcal{Q} \otimes \mathcal{S}^*).$$

We denote the maps of sheaves given by the generators by: $\hat{d}_1^*, \hat{d}_2^*$ and $\hat{a}_3$.

Consider the sheaf

$$\mathcal{O}_{Z'} = Sym(\mathcal{R}^* \otimes Sym(\mathcal{Q} \otimes \mathcal{S}^*) \otimes Sym(\mathcal{O}(1))$$

whose sections have additionally the representation $\tilde{a}_3$, so far with no relations connecting it to d_2^*, d_1^*.

Finally we consider the factor sheaf

$$\mathcal{O}_Z = \oplus_{a,\lambda,b}\ S_a\mathcal{R}^* \otimes S_\lambda\mathcal{Q} \otimes S_\lambda\mathcal{S}^* \otimes \mathcal{O}(b)/(\mathcal{I})$$

where we sum over the partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$, and $\mathcal{I}$ is an ideal sheaf generated by the subsheaf which can be viewed as identifying representations $\bigwedge^{n-1} Q \otimes \bigwedge^{n-1} \mathcal{S}^*$ and $\mathcal{R}^* \otimes \mathcal{O}(2)$. The identification is $SL(F) \times Spin(G)$-equivariant.

Taking the cohomology of $\mathcal{O}_Z$ we see that higher cohomologies vanish and we get

$$A(n)_1 = H^0(Grass(1, F) \times IGrass(n-1, G), \mathcal{O}_Z) =$$

$$= \oplus_{a,\lambda,b} S_{\lambda_1,\ldots,\lambda_{n-2},0,-a} F \otimes V((\lambda_1 - \lambda_2)\omega_1 + \ldots + (\lambda_{n-3} - \lambda_{n-2})\omega_{n-3} + \lambda_{n-2}(\omega_{n-2} + \omega_{n-1}) + b\omega_{n-2}; D_{n-1}).$$

Here we sum over $a, b \ge 0$ and partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-2})$.

The structure sheaf of $\mathcal{O}_Z$ has a decomposition and the sections of $\mathcal{I}$ give exactly the relation described in the previous section.

By construction over the ring $A(n)_1$ we have a selfdual complex $\mathbb{F}_\bullet^{A(n)_1}$ of length 4:

$$\mathbb{F}_\bullet^{A(n)_1} : 0 \to H\otimes A(n)_1 \to F\otimes_K A(n)_1 \to G\otimes_K A(n)_1 \cong G^*\otimes_K A(n)_1 \to F^*\otimes_K A(n)_1 \to H^*\otimes A(n)_1$$

with the maps given by ϕ, ψ, ψ^* and ϕ^* respectively.

For any $A(n)_1$-algebra B we will denote

$$\mathbb{F}_\bullet^B := \mathbb{F}_\bullet^{A(n)_1} \otimes_{A(n)_1} B.$$

We will also use certain natural complexes associated to $\mathbb{F}_\bullet^{A(n)_1}$. One of them is

$$\mathbb{F}_\bullet'^{A(n)_1} :\ 0 \to S_2H \otimes A(n)_1 \to (H \otimes F \otimes A(n)_1) \otimes A(n)_1 \to \bigwedge^2 F \otimes A(n)_1 \to \bigwedge^2 G \otimes A(n)_1.$$

Likewise, for any $A(n)_1$-algebra B we will denote

$$\mathbb{F}_\bullet'^B := \mathbb{F}_\bullet'^{A(n)_1} \otimes_{A(n)_1} B.$$

Our next step is to describe the homology modules of the complex $\mathbb{F}_\bullet^{A(n)_1}$. We start by counting representations in each term of the complex in each $SL(F) \times Spin(G)$-isotypic component.

Note that the whole situation is $\mathbb{Z}_2$-graded, i.e. it decomposes according to parity of b. The same is true for the homology modules of $\mathbb{F}_\bullet^{A(n)_1}$.

Counting representations, using Klimyk's formula to calculate $G\otimes V(\lambda)$ for representations of $Spin(2n-2)$ we get the following.

Proposition 4.1. *(1) The homology modules $H_4(\mathbb{F}_\bullet^{A(n)_1})$, $H_3(\mathbb{F}_\bullet^{A(n)_1})$ are zero,*

(2) The homology module $H_2(\mathbb{F}_\bullet^{A(n)_1})$ is generated by the cycle

$$q_1 : V(\omega_{n-1}) \otimes A(n)_1 \to G \otimes A(n)_1,$$

which sends the representation $V(\omega_{n-1})$ to G times the summand corresponding to $b = 1$, $\lambda = 0, a = 0$. Explicitly, the basis element w_I in $V(\omega_{n-1})\otimes A(n)_1$ corresponding to the subset I of $[1, n-1]$ of odd cardinality is sent to

$$\sum_{i=1}^{n-1} e_i \otimes \tilde{a}_{3,I\cup\{i\}} + \sum_{I=1}^{n-1} e_i^* \otimes \tilde{a}_{3,I\setminus\{i\}}.$$

(3) The homology module $H_1(\mathbb{F}_\bullet^{A(n)_1})$ is generated by representation $\bigwedge^2 F^*$ giving the Koszul relations on the entries of the matrix ϕ.

5. Higher structure maps p_i.

Note that the lifting p_1 is not unique. It can be modified by any map from $V(\omega_{n-1}) \otimes A(n)_1 \to H \otimes A(n)_1$. We denote this defect by

$$\mathbb{L}_1 = V(\omega_{n-1})^*.$$

We will use the method employed in [58] in the case of resolutions of length 3. It consists of finding more of such cycle factorizations (called higher structure theorems). After adding the coefficients of factorization p_1 we modify the ring $A(n)_1$ dividing by relations satisfied by all possible choices of p_1 and taking strict transform with respect to the ideal of entries of d_4. We call the ring obtained in this way $A(n)_2$. Our goal is to keep doing it until the homology H_2 of complex $\mathbb{F}_\bullet^{(1)}$ extended to the bigger ring is equal to zero.

Right now we look at what happens over the ring $A(n)_2$. The next step in our construction is as follows.

We consider the complex $\mathbb{F}'^{A(n)_2}_\bullet$ associated to the complex $\mathbb{F}^{A(n)_2}_\bullet$. It is a complex

$$\mathbb{F}'^{A(n)_2}_\bullet:\ 0 \to S_2H \otimes A(n)_2 \to (H \otimes F) \otimes A(n)_2 \to \bigwedge^2 F \otimes A(n)_2 \to \bigwedge^2 G \otimes A(n)_2$$

The first two maps are induced by d_4 and the third one is the second exterior power of d_3.

Proposition 5.1. *Consider the map*

$$q_2 : \mathbb{L}_2^* \subset \bigwedge^2 V(\omega_{n-1}) \otimes A(n)_2 \to \bigwedge^2 F \otimes A(n)_2.$$

where $\mathbb{L}_2$ *is a factor*

$$0 \to V(\omega_{n-3}) \to \bigwedge^2 V(\omega_{n-1})^* \to \mathbb{L}_2 \to 0.$$

Then the image of q_2 *is the cycle in the complex* $\mathbb{F}'_\bullet$.

We introduce (in analogy with [58]) the defect graded Lie algebra

$$\mathbb{L}_\bullet = \oplus_{i\ge 1}\mathbb{L}_i$$

which is generated by $\mathbb{L}_1$ and defined by quadratic relations given by the kernel K

$$0 \to K \to \bigwedge^1 \mathbb{L}_1 \to \mathbb{L}_2 \to 0.$$

The defect Lie algebra has another interpretation in terms of E_n root systems. These, in terminology of [58] are the graphs $T_{n-3,2,3}$. We consider the grading on the Kac-Moody Lie algebra $\underline{g}(T_{n-3,2,3})$ defined by the simple root corresponding to the node z_2. Graphically we have

$$\begin{array}{ccccccccc} x_{n-4} & - & x_{n-5} & \dots & x_1 & - & u & - & y_1 \\ & & & & & & | & & \\ & & & & & & z_1 & & \\ & & & & & & | & & \\ & & & & & & z_2 & & \end{array}$$

Proposition 5.2. *The algebra* $\mathbb{L}_\bullet$ *is the positive part of the Kac-Moody Lie algebra* $\underline{g}(T_{1,n-3,3})$ *in the grading defined by the simple root corresponding to the node* z_2.

Proof. This is based on parabolic form of Kostant's theorem on cohomology of positive part of a Kac-Moody Lie algebra and is similar to analogous fact for $T_{p,q,r}$ case (see [58]). □

Recall that the graded components of the Lie algebra $\mathbb{L}_\bullet$ can be defined by the exact sequence

$$0 \to \mathbb{L}_{i+1}^* \to (\bigwedge^2 \mathbb{L}_\bullet)_{i+1}^* \to (\bigwedge^3 \mathbb{L}_\bullet)_{i+1}^*$$

We start with the cycle p_2. Let

$$0 \to H \otimes R \to F \otimes R \to G \otimes R = G^* \otimes R \to F^* \otimes R \to H^* \otimes R$$

be a finite free resolution of a cyclic module R/I where I is a Gorenstein ideal of codimension 4. The differentials are denoted: d_4, d_3, d_3^t, d_4^t. We will consider the associated complexes

$$\mathbb{F}'_\bullet : 0 \to S_2H \otimes R \to H \otimes F \otimes R \to \bigwedge^2 F \otimes R \to \bigwedge^2 G \otimes R$$

The first two differentials (from the left) are induced by d_4, the last one is $\bigwedge^2 d_3$. We will finally consider the complex

$$\mathbb{F}''_\bullet : 0 \to S_3H \otimes R \to S_2H \otimes F \otimes R \to H \otimes \bigwedge^2 F \otimes R \to \bigwedge^3 F \otimes R,$$

The beginning of the Koszul complex on d_4.

Proposition 5.3. *(1) We have the cycle $q_2 : \mathbb{L}_2^* \to \bigwedge^2 F \otimes R$ in the complex $\mathbb{F}'_\bullet$. Its lifting p_2 has defect $\mathbb{L}_2$.*

(2) For each $i \geq 3$ we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & S_3H \otimes R & \to & S_2H \otimes F \otimes R & \to & H \otimes \bigwedge^2 F \otimes R & \to & \bigwedge^3 F \otimes R \\ & & & & \uparrow\ p_{i+1} & & \uparrow\ \sum(p_a \wedge p_b) & & \uparrow\ \sum(p_a \wedge p_b \wedge p_c) \\ & & 0 & \to & \mathbb{L}_{i+1}^* & \to & (\bigwedge^2 \mathbb{L}_\bullet)_{i+1}^* & \to & (\bigwedge^3 \mathbb{L}_\bullet)_{i+1}^* \end{array}$$

Which gives us a map with defect $\mathbb{L}_{i+1}$.

Proof. The second part follows from acyclicity of the upper row and from the fact that the right square is obviously commutative. To prove the second part we notice that by localization it is enough to prove that q_2 is a cycle for a split self-dual complex of the format $(1, n, 2n-2, n, 1)$. We do it as follows. We set the bases of H, F and of G to be $\{h\}$, $\{f_1, \ldots, f_n\}$, $\{e_1, \ldots, e_{n-1}, e_{n-1}^*, \ldots, e_1^*\}$ respectively. We set $d_4(h) = f_n$, $d_3(f_i) = e_i$ for $1 \leq i \leq n-1$, $d_3(f_n) = 0$. The quadratic form on G is the duality. We have the spinor coordinates $(\tilde{a}_3)_\emptyset = 1$, all other $(\tilde{a}_3)_I$ for non-empty even cardinality subsets I equal to 0.

Consider the map $V(\omega_{n-1}) \to G \otimes V(\omega_{n-2})$ described in section 2.2. The map q_1 sends the basis vector w_I ($I \subset [1, n-1]$, $|I|$ odd) to

$$\sum_i^{n-1} (\tilde{a}_3)_{I \setminus i} e_i^* + \sum_{i=1}^{n-1} (\tilde{a}_3)_{I \cup i} e_i.$$

The lifting p_i sends w_i to $f_i + t_i f_n$ (for some $t_i \in R$). For I of cardinality ≥ 3 $p_1(W_I) = t_I f_n$ for some $t_I \in R$. Now we are ready to prove that the image of the map q_2 is contained in the kernel of the differential of $\mathbb{F}'_\bullet$. Its composition with the second exterior power of d_3 can be possibly non-zero only on the decomposable vectors $w_i \wedge w_j$, otherwise we will have a factor f_n in each summand so the image will be zero. But the weights of $w_i \wedge w_j$ are only occurring in $V(\omega_{n-3})$, as they are in one Weyl orbit, and it is the orbit of the highest weight vector in $\bigwedge^2 V(\omega_{n-1})$. This concludes the proof of the proposition. □

6. The E_n program

In this section I continue to develop the theory of generic ring for the complex $\tilde{\mathbb{F}}_\bullet$ which is a complex of length 3 we obtain from $\mathbb{F}_\bullet$ by omitting the term at the right end. I use the approach similar to that of [58]. The resulting ring $A(n)_\infty$ has a multiplicity free action of the Lie algebra $\underline{s}l_n \times \underline{g}(T_{n-3,2,3})$. Its structure allows also to define the opposite complex $\mathbb{F}_\bullet^{top}$ which points out to the structure of the open subset U_{Gor} of points in$Spec(A(n)_\infty)$ for which the selfdual complex $\mathbb{F}_\bullet^{A(n)_\infty}$ defines a selfdual acyclic complex of length 4.

6.1. **The ring $A(n)_\infty$.** We defined the sequence of factorizations p_i with defects $\mathbb{L}_i$ for $i \geq 1$. We define inductively a sequence of rings $A(n)_i$. Assume we defined the ring $A(n)_i$ that contains the entries of all structure maps $p_1, p_2, \ldots, p_i$. To construct $A(n)_{i+1}$ we proceed as follows. First we add to $A(n)_i$ the entries of the map p_i and its factorization relation. Then we also divide by the relations satisfied by entries of all liftings p_i. Finally we factor out annihilators of ideals $I(d_3)$ and $I(d_4)$ maximal non vanishing minors of d_4 and d_3, and we take the ideal transforms with respect to $I(d_3)I(d_4)$. The union of rings $A(n)_i$ is called $A(n)_\infty$.

Theorem 6.1. *(1) The ring $A(n)_\infty$ is a generic ring for complexes of length 3 of formats $(n, 2n-2, n, 1)$ where the module of rank $2n-2$ is orthogonal and first and second differential are transpose to each other. This means that the complex $\mathbb{F}_\bullet^{A(n)_\infty}$ has only non-zero homologies H_1 and H_0.*

(2) The ring $A(n)_\infty$ has a multiplicity free decomposition to the representations of Lie algebra $\underline{s}l_n \times \underline{g}(T_{n-3,2,3})$, namely

$$A(n)_\infty = \oplus_{\lambda,a,b} S_{(\lambda_1,\lambda_2,\ldots,\lambda_{n-2},0,-a)}F \otimes V(\mu(\lambda, a, b))$$

where the weight $\mu(\lambda, a, b)$ is given by labeling of $T_{n-3,2,3}$ as follows

$$\mu(\lambda,a,b) = \begin{array}{ccccccccc} (\lambda_1-\lambda_2) & - & (\lambda_2-\lambda_3) & \ldots & (\lambda_{n-4}-\lambda_{n-3}) & - & (\lambda_{n-3}-\lambda_{n-2}) & - & (b+\lambda_{n-2}) \\ & & & & & & | & & \\ & & & & & & \lambda_{n-2} & & \\ & & & & & & | & & \\ & & & & & & a & & \end{array}$$

(3) The ring $A(n)_\infty$ is Noetherian if and only if $n \leq 8$.

Proof. We procced in five steps

1. Prove that the Lie algebra $\mathbb{L}_\bullet$ acts on the ring $A(n)_\infty$ by derivations,
2. Identify defect variables with the coordinates on the big cell N in the homogeneous space $G(T_{2,n-3,3})/P_{z_2}$ (P_{z_2} denotes the parabolic subgroup corresponding to the node z_2).,
3. The subring $B(n)$ generated by the action of $\mathbb{L}$ from $A(n)_1$ gives a subalgebra of $A(n)_\infty$ with the multiplicity free action of $G(T_{2,n-3,3)} \times SL(F)$ with the decomposition given by the theorem,
4. The structure maps p_i are visible in the isotypic component corresponding to $\lambda = a = 0, b = 1$,
5. Identify $B(n)$ with $A(n)_\infty$.

We start with the step (1).

Proposition 6.2. *The Lie algebra $\mathbb{L}_\bullet = \oplus_{i\geq 1}\mathbb{L}_i$ acts on the ring $A(n)_\infty$ by derivations.*

Proof. Let us denote $\mathbb{L}_\bullet^{(i)}$ the nilpotent Lie algebra

$$\mathbb{L}_\bullet^{(i)} = \oplus_{j=1}^{i}\mathbb{L}_i.$$

Recall that $A(n)_\infty$ is constructed by induction, as a limit of the rings $A(n)_i$ and that $A(n)_{i+1}$ is constructed from $A(n)_i$ by a three step process.

(1) Add entries of the matrix factorization p_i,
(2) divide by relations satisfied by all liftings p_i of the cycle q_i,
(3) take ideal transform with respect to the ideal $I(d_4)I(d_3)$.

We first note that in the open sets $U(I(d_4)I(d_3))$ in $Spec(A(n)_i)$ we deal with a split complex and there we just add defect variables to the ring $A(n)_1$. Let us denote by K_i the field of fractions of $A(n)_i$. This means that

$$A(n)_\infty \otimes_{A(n)_1} K_1 = K_1 \times Sym(\mathbb{L}_\bullet).$$

The spectrum of $A(n)_\infty \otimes_{A(n)_1} K_1$ is therefore the coordinate ring of the big cell in the homogeneous space $G(T_{2,n-3,3})/P_{z_2}$ (P_{z_2} denotes the parabolic subgroup corresponding to the node z_2). Therefore the Lie algebra $\mathbb{L}_\bullet$ acts on this big cell, so it acts on $A(n)_\infty \otimes_{A(n)_1} K_1$ by derivations. We will show by induction on i that this action descends to the action of $\mathbb{L}_\bullet^{(i)}$ on $A(n)_{i+1}$.

For $i=1$ it is clear as the commutative Lie algebra $\mathbb{L}_1$ clearly acts on $A(n)_2$.

Assume we constructed the action of $\mathbb{L}^{(i)}$ on $A(n)_{i+1}$. Note that over $A(n)_{i+1}$ we have $depth(I(d_4)) \geq 2$, because ideal transform raises canonically the depth from 1 to 2 (see [58], section 5).

This means the Koszul complex of $I(d_4)$ is exact at the first two left-most terms:

$$0 \to S_3H \otimes A(n)_i \to S_2H \otimes F \otimes A(n)_i \to H \otimes \bigwedge^2 F \otimes A(n)_i.$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & S_3H \otimes A(n)_i & \to & S_2H \otimes F \otimes A(n)_i & \to & H \otimes \bigwedge^2 F \otimes A(n)_i & \to & \bigwedge^3 F \otimes A(n)_i \\
 & & & & \uparrow\ p_{i+1} & & \uparrow\ \sum(p_a \wedge p_b) & & \uparrow\ \sum(p_a \wedge p_b \wedge p_c) \\
 & & 0 & \to & \mathbb{L}^*_{i+1} & \to & (\bigwedge^2 \mathbb{L}_\bullet)^*_{i+1} & \to & (\bigwedge^3 \mathbb{L}_\bullet)^*_{i+1}
\end{array}$$

Applying the derivation D to the left commuting square in the diagram we see that for any derivation D such that $A(n)_1$ is contained in the constants of D, the values of D on entries of p_{i+1} are completely determined. It applies to any derivation of elements of $\mathbb{L}_\bullet^{(i)}$. So these derivations descend to $A(n)_\infty \otimes_{A(n)_1} K_1$. This means that all derivations from $\mathbb{L}^{(i+1)}$ have values in $A(n)_{i+1} \otimes_{A(n)_{i+1}} K_{k+1}$. It remains to see that these derivations take an ideal transform with respect to $I(d_4)I(d_3)$ to itself. But it follows from the quotient rule for derivations and from the fact that generators of $I(d_4)I(d_3)$ are constants with respect to all derivations from $\mathbb{L}_\bullet$. $\square$

Step (2) is straightforward as defect variables are exactly the roots of $\underline{g}(T_{n-3,2,3})$ which are not in the parabolic subalgebra of P_{z_2}. To prove step (3) we note that the summand of $B(n)$ corresponding to the triple (λ, a, b) is $S_{\lambda_1,\ldots,\lambda_{n-2},0,a)}F$ tensored with the lowest weight module generated by the corresponding representation of $\underline{so}(2n-2)$. But this representation has a dominant weight, so the induced module is the corresponding irreducible representation of $T_{n-3,2,3}$.

Step (4) is then straightforward, as the 0-th graded component of the corresponding representation of $T_{(n-3,2,3)}$ is just $\mathbb{C}$, so in the i-th graded component we will see $\mathbb{L}_i$.

The inclusion $B(n) \to A(n)_\infty$ becomes an isomorphism after inverting any entry of d_4, as both rings become just the corresponding localizations of $A(n)_1$ extended by defect variables. Let $I = (x_1, \ldots, x_n)$ be the ideal of d_4. It is clearly contained in B. Its depth is ≥ 2 because otherwise these elements would have a common factor but they do not have it in $A(n)_\infty$.

To prove the formula for the decomposition of $A(n)_\infty$ we notice that the strict transform is defined as $j_{\infty *}(\mathcal{O}_{U_\infty})$ where U_∞ is the complement of $V(I(d_4))$, and $j_\infty : U_\infty \to Spec(A_\infty)$ is a natural embedding. in $Spec(A(n)_\infty)$. But this is exactly the open cell in $G(T_{2,n-3,3})/P_{z_2}$. Decomposing $j_*(\mathcal{O}_{U_\infty})$ to $SL(F)$-isotypic components , we see that in each $SL(F)$-isotypic component of $S_\mu F$ we just see the kernel of the parabolic Grothendieck-Cousin complex on $G(T_{2,n-3,3})/P_{z_2}$ corresponding to the weight corresponding to μ given in the theorem. So this kernel is the irreducible representation given in the theorem. □

The theorem then follows from the next proposition.

Proposition 6.3. *Let us preserve the notation of previous theorem.*

(1) The sheaf $R^1 j_{\infty *}(\mathcal{O}_{U_\infty})$ is zero.

(2) The ring $A(n)_\infty$ with respect to the truncated complex $\mathbb{F}_\bullet^{A(n)_\infty}$, i.e. $\mathbb{F}_\bullet^{A(n)_\infty}$ has only non-zero homologies H_1 and H_0.

Proof. The first fact follows from properties of Borel-Weyl-Bott theorem. More precisely, we see that the decomposition of $\mathcal{O}_{U_\infty}$ to coherent sheaves is

$$\mathcal{O}_{U_\infty} = \oplus_{\lambda,a,b} S_{(\lambda_1,\lambda_2,\ldots,\lambda_{n-2},0)}\mathcal{Q} \otimes S_a\mathcal{R}^* \otimes \mathcal{V}(\mu(\lambda,a,b))$$

where $\mathcal{V}(\mu(\lambda,a,b))$ is a sheaf on our homogeneous space $G(T_{2,n-3,3})/P_{z_2}$ corresponding to $(\mu(\lambda,a,b))$ and a can take arbitrary integer values. More precisely, the sheaf $\mathcal{V}(\mu(\lambda,a,b))$ is a pushdown of a line bundle $\mathcal{L}(\mu(\lambda,a,b))$ on $G(T_{2,n-3,3})/B$ under the natural projection.

We denote the summand corresponding to (λ,a,b) in the above formula by $\mathcal{M}(\lambda,a,b)$.

This means that the higher direct image $R^1 j_{\infty *}(\mathcal{O}_{U_\infty})$ is a direct sum of cohomology modules $H^1\mathcal{M}(\lambda,a,b)$ of the sheaf on the homogeneous space $Grass(n-1,F) \times G(T_{2,n-3,3})/P_{z_2}$. So it is enough to show that the cohomology group H^1 of any bundle $\mathcal{M}(\lambda,a,b)$ with a arbitrary integer is zero. But this is obvious by Bott theorem since for a nonnegative our bundle has sections, and for a negative we have to go through some reflection on both factors to reach the dominant weight.

To see why the first statement of the proposition implies the second, see [58] section 5. □

After adding to $A(n)_1$ the entries of factorizations p_i lifting the cycles q_i, dividing by the relations satisfied by all the liftings, and taking ideal transform with respect to $I(d_3)$, $I(d_4)$ we obtain the ring $A(n)_\infty$ with the action of Lie algebra

$$\underline{g}(T_{2,n-3,3}) \times \underline{gl}(F).$$

The next step is to investigate the open set U_{Gor} in $Spec(A(n)_\infty)$ of points where the complex $\mathbb{F}_\bullet^{A(n)_\infty}$ is acyclic, i.e. the complement of the support of $H_1(\mathbb{F}_\bullet^{A(n)_\infty})$.

Analyzing the generators of the ring $A(n)_\infty$ similarly to [59] we see that the decomposition of $A(n)_\infty$ suggests that its generators will come from three representations. They are: W_1

corresponding to $\lambda = (1)$, $a = 0$, $b = 0$, W_{n-1} corresponding to $\lambda = 0$, $a = 1$, $b = 0$ and W_0 corresponding to $\lambda = 0$, $a = 0$, $b = 1$. We call them *the critical representations.*

Each of these representations acquires the grading induced by the grading on $\underline{g}(T_{2,n-3,3})$.

We will use the following notation. The lowest graded component will by convention be in degree 0. The graded components of W_1 will be denoted by $v_i^{(1)}$, the graded components of W_{n-1} will be denoted by $v_i^{(n-1)}$ and the graded components of W_0 will be denoted by $v_i^{(0)}$.

Analyzing these graded components is an important task that allows to better understand what is going on.

6.2. **Small cases:** $n = 4$**,** $n = 5$**.**

6.2.1. *Case* $n = 4$. The ring $A(4)_\infty$ decomposes as follows

$$A(4)_\infty = \oplus_{(\lambda_1,\lambda_2),a,b} S_{(\lambda_1,\lambda_2,0,-a)}F \otimes V(\mu(\lambda, a, b),$$

Where

$$\mu(\lambda,a,b) = \begin{array}{ccc} (\lambda_1-\lambda_2) & - & (\lambda_2+b) \\ | & & \\ \lambda_2 & & \\ | & & \\ a & & \end{array}.$$

for the diagram $E_4 = A_4$. We denote corresponding 5-dimensional space by U'. The grading on $\underline{sl}(U')$ corresponds to decomposition $U' = U \oplus K$. The orthogonal space G becomes $\bigwedge^2 U$, with U and $\bigwedge^3 U$ being two half-spinor representations (because the corresponding graph is $D_3 = A_3$). The graded components of critical representations are:

$$W_1 = F \otimes (\bigwedge^2 U \oplus U),$$
$$W_3 = F^* \otimes (\bigwedge^4 U \oplus \bigwedge^3 U),$$
$$W_0 = U \oplus K.$$

We have $v_0^{(1)} = d_3$, $v_0^{(3)} = d_4$, $v_1^{(3)} = p_1$, $v_0^{(0)} = \tilde{a}_3$. The meaning of $v_1^{(1)}$ is given below for general n.

The generic Gorenstein ideals of codimension 4 with 4 generators are just complete intersections. This is equivalent to the map p_1 being an isomorphism. The open set U_{Gor} is given by the condition $det(p_1) \neq 0$. This observation is important because it indicates looking at the top components of critical representations is the key.

6.2.2. *Case* $n = 5$. The ring $A(5)_\infty$ decomposes as follows

$$A(5)_\infty = \oplus_{(\lambda_1,\lambda_2,\lambda_3),a,b} S_{(\lambda,\lambda_2,\lambda_3,0,-a)}F \otimes V(\lambda, a, b),$$

where

$$\mu(\lambda,a,b) = \begin{array}{ccccc} (\lambda_1-\lambda_2) & - & (\lambda_2-\lambda_3) & - & (\lambda_3+b) \\ & & | & & \\ & & \lambda_3 & & \\ & & | & & \\ & & a & & \end{array}.$$

for the diagram $E_5 = D_5$.

We denote corresponding 10-dimensional orthogonal space by G'. The grading on $\underline{s}o(G')$ corresponds to decomposition $G' = K \oplus G \oplus K^*$. The orthogonal space G is the one occurring in our resolution. The decompositions in hyperbolic bases are denoted $G' = W' \oplus W'^*$ and $G = W \oplus W^*$. The graded components of critical representations are:

$$W_1 = F \otimes (W' \oplus \bigwedge^3 W' \oplus \bigwedge^5 W'),$$
$$W_4 = F^* \otimes (W' \oplus W'^*),$$
$$W_0 = \mathbb{C} \oplus \bigwedge^2 W' \oplus \bigwedge^4 W'.$$

We have $v_0^{(1)} = d_3$, $v_0^{(4)} = d_4$, $v_1^{(4)} = p_1$, $v_0^{(0)} = \tilde{a}_3$. The meaning of $v_1^{(1)}$ is given below for general n.

Let us perform the computation in the split exact case. We start with the split exact complex, with the basis of F being $\{f_1, \ldots, f_5\}$, basis of G being $\{\bar{e}_1, \ldots, \bar{e}_4, e_4, \ldots, e_1\}$. The matrices of the differentials are

$$d_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map $\tilde{a}_3 : R \to V(\omega_4, D_5)$ sends $1 \mapsto u_\emptyset$. So $(\tilde{a}_3)_\emptyset = 1$, $(\tilde{a}_3)_I = 0$ for $I \neq \emptyset$. Since the cycle q_1 is given by the formula

$$u_I \mapsto \sum_{i=1}^{4} (\tilde{a}_3)_{I \cup \{i\}} e_i + \sum_{i=1}^{4} (\tilde{a}_3)_{I \setminus \{i\}} \bar{e}_i$$

we get that $q_1 : V(\omega_5, D_5) \to G$ sends $u_{\{i\}}$ to $\bar{e}_i$ for $1 \leq i \leq 4$ and sends u_I to 0 for $|I| > 1$. This means that

$$p_1(u_{\{i\}}) = f_i + b_{\{i\}} f_5$$

for $1 \le i \le 4$ and

$$p_1(u_I) = b_I f_5$$

for $|I| > 1$.

We can calculate $v_2^{(4)}$ using relations of degree $(0,2,0)$ and we get the matrix

$$v_2^{(4)} = \begin{pmatrix} b_{234} \\ -b_{134} \\ b_{124} \\ -b_{123} \\ b_1 b_{234} - b_2 b_{134} + b_3 b_{124} - b_4 b_{123} \end{pmatrix}$$

Similarly, calculating $v_2^{(1)}$ using the relations of degree $(2,0,0)$ we get the matrix

$$v_2^{(1)} = \begin{pmatrix} b_{134} & -b_{124} & b_{123} & 0 & 0 & 0 & b_1 & -b_1 \\ b_{234} & 0 & 0 & b_{124} & -b_{123} & 0 & b_2 & -b_2 \\ 0 & b_{234} & 0 & b_{134} & 0 & b_{123} & b_3 & -b_3 \\ 0 & 0 & b_{234} & 0 & b_{134} & b_{124} & b_4 & -b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Here the order of columns is $\{1, 2, 3, \bar{3}, \bar{2}, \bar{1}, 4, \bar{4}\}$.

Note that the syzygy matrix of the transpose of $v_2^{(1)}$ is

$$\begin{pmatrix} b_{234} \\ -b_{134} \\ b_{124} \\ -b_{123} \\ b_1 b_{234} - b_2 b_{134} + b_3 b_{124} - b_4 b_{123} \end{pmatrix}$$

So the complex $\mathbb{F}^{top}$ is just a Gorenstein resolution of a complete intersection of codimension 4 plus a split complex with Betti numbers $(0,1,2,1,0)$. The generic Gorenstein ideals of codimension 4 with 5 generators are, by Kunz's theorem, just non minimal resolutions of complete intersections. The complex $\mathbb{F}^{top}$ for the split exact complex gives just that.

6.3. **The E_6, E_7, E_8 triplets.** These three cases are separate from the others because one can apply the idea of constructing the complex $\mathbb{F}^{top}_\bullet$ from [59]. They turn out to be uniformly related to the codimennsion 4 Gorenstein Schubert varieties in the homogeneous spaces $G(E_n)/P_1$ for $n = 6, 7, 8$. In the sequel we will denote Z_n $(n = 6, 7, 8)$ the big open cell in the homogeneous space $G(E_n)/P_1$.

6.3.1. *Case $n = 6$.* The ring $A(6)_\infty$ decomposes as follows

$$A(6)_\infty = \oplus_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4),a,b} S_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,0,-a)} F \otimes V(\mu(\lambda, a, b)),$$

where

$$\mu(\lambda, a, b) = \begin{matrix} (\lambda_1-\lambda_2) & - & (\lambda_2-\lambda_3) & - & (\lambda_3-\lambda_4) & - & (\lambda_4+b) \\ & & & & | & & \\ & & & & \lambda_4 & & \\ & & & & | & & \\ & & & & a & & \end{matrix} .$$

Let us look at the decompositions of critical representations. We have

$$W_1 = F \otimes (G \oplus V(\omega_4, D_5) \oplus \mathbb{C}),$$
$$W_5 = F^* \otimes (\mathbb{C} \oplus V(\omega_5, D_5) \oplus G),$$
$$W_0 = V(\omega_5, D_5) \oplus [\underline{so}(10) \oplus \mathbb{C}] \oplus V(\omega_4, D_5).$$

Notice that because of the highest-lowest weight duality the top components $v_2^{(1)}$ and $v_2^{(5)}$ can be arranged in two differentials of another self-dual complex. By analogy with [59] we have

Conjecture 6.4. *The open set U_{Gor} is equal to the set U_{split} of points where the complex $\mathbb{F}_\bullet^{top}$ is split exact.*

Here we will prove a partial result

Theorem 6.5. *We have $U_{split} \subset U_{Gor}$.*

Proof. We apply the technique of reverse calculation by starting with the split exact complex, and calculating higher structure theorems $v_2^{(1)}$ and $v_2^{(5)}$ for this complex with the generic defect variables. Then we prove that the resulting self-dual complex of length 4 over a polynomial ring $S := \mathbb{C}[Z_6]$ on the defect variables is a resolution of a cyclic S-module S/J_6 where J_6 is a Gorenstein ideal of codimension 4 with 6 generators. $\square$

Looking more closely at the graded components of critical representations we get:

$$v_0^{(1)} = d_3,$$
$$v_0^{(5)} = d_4, v_1^{(5)} = p_1,$$
$$v_0^{(0)} = \tilde{a}_3.$$

Let us perform the computation in the split exact case. We start with the split exact complex, with the basis of F being $\{f_1, \dots, f_6\}$, basis of G being $\{\bar{e}_1, \dots, \bar{e}_5, e_5, \dots, e_1\}$. The matrices of the differentials are

$$d_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} 0&0&0&0&0&1 \end{pmatrix}.$$

The map $\tilde{a}_3 : R \to V(\omega_4, D_5)$ sends $1 \mapsto u_\emptyset$. So $(\tilde{a}_3)_\emptyset = 1$, $(\tilde{a}_3)_I = 0$ for $I \neq \emptyset$. Since the cycle q_1 is given by the formula

$$u_I \mapsto \sum_{i=1}^{5} (\tilde{a}_3)_{I\cup\{i\}} e_i + \sum_{i=1}^{5} (\tilde{a}_3)_{I\setminus\{i\}} \bar{e}_i$$

we get that $q_1 : V(\omega_5, D_5) \to G$ sends $u_{\{i\}}$ to $\bar{e}_i$ for $1 \le i \le 5$ and sends u_I to 0 for $|I| > 1$. This means that

$$p_1(u_{\{i\}}) = f_i + b_{\{i\}} f_6$$

for $1 \le i \le 5$ and

$$p_1(u_I) = b_I f_6$$

for $|I| > 1$.

Using relations of degree $(0,2,0)$ we calculate the map $v_2^{(5)}$ as a map from G to F. Let us order the columns to be $\{1,2,3,4,5,\bar{5},\bar{4},\bar{3},\bar{2},\bar{1}\}$. The matrix we get is

$$v_2^{(5)} = \begin{pmatrix} 0 & -b_{345} & b_{245} & -b_{235} & b_{234} & 0 & 0 & 0 & 0 & -b_{12345}\\ b_{345} & 0 & -b_{145} & b_{135} & -b_{134} & 0 & 0 & 0 & -b_{12345} & 0\\ -b_{245} & b_{145} & 0 & -b_{125} & b_{124} & 0 & 0 & -b_{12345} & 0 & 0\\ b_{235} & -b_{135} & b_{125} & 0 & -b_{123} & 0 & -b_{12345} & 0 & 0 & 0\\ -b_{234} & b_{134} & -b_{124} & b_{123} & 0 & -b_{12345} & 0 & 0 & 0 & 0\\ C_1 & -C_2 & -C_3 & -C_4 & C_5 & \bar{C}_5 & -\bar{C}_4 & \bar{C}_3 & -\bar{C}_2 & \bar{C}_1 \end{pmatrix}$$

where

$$C_1 = b_2 b_{345} - b_3 b_{245} + b_4 b_{235} - b_5 b_{234},$$

$$\bar{C}_1 = b_{123} b_{145} - b_{124} b_{135} + b_{125} b_{134} - b_1 b_{12345},$$

and analogously for C_i and $\bar{C}_i$.

Denoting R_i the i-th row of this matrix and calculating

$$R_6 - \sum_{i=1}^{5} b_i R_i$$

we get

$$\big(0,0,0,0,0,Pf(5),Pf(4),Pf(3),Pf(2),Pf(1)\big)$$

where $Pf(i)$ is the (signed) 4×4 Pfaffian of the upper left 5×5 block of the above matrix obtained by removing the i-th row and column of this submatrix. After this operation we get the syzygy matrix of the ideal generated by the above Pfaffians and b_{12345}, i.e. hyperplane section in the generic codimension 3 Pfaffian ideal.

It remains to calculate the component $v_2^{(1)}$. By using relations of degree $(2,0,0)$ we see that it is the matrix

$$\big(Pf(5), Pf(4), Pf(3), Pf(2), Pf(1), b_{12345} - \sum_{i=1}^{5} b_i Pf(i)\big)$$

6.3.2. *Case $n=7$.* The ring $A(7)_\infty$ decomposes as follows

$$A(7)_\infty = \oplus_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5),a,b} S_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,0,-a)} F \otimes V(\mu(\lambda,a,b)),$$

where

$$\mu(\lambda,a,b) = \begin{matrix} (\lambda_1-\lambda_2) & - & (\lambda_2-\lambda_3) & - & (\lambda_3-\lambda_4) & - & \lambda_4-\lambda_5 & - & (\lambda_5+b) \\ & & & & & & | & & \\ & & & & & & \lambda_5 & & \\ & & & & & & | & & \\ & & & & & & a & & \end{matrix}.$$

Let us look at the decompositions of critical representations. We have

$$W_1 = F \otimes (G \oplus V(\omega_6, D_6) \oplus G),$$

$$W_6 = F^* \otimes (\mathbb{C} \oplus V(\omega_5, D_6) \oplus [\underline{so}(12) \oplus \mathbb{C}] \oplus V(\omega_5, D_6) \oplus \mathbb{C}),$$

$$W_0 = V(\omega_6, D_6) \oplus [V(\omega_1, D_6) \oplus V(\omega_3, D_6)] \oplus [V(\omega_1+\omega_6, D_6) \oplus V(\omega_5, D_6)] \oplus$$
$$\oplus [V(\omega_1, D_6) \oplus V(\omega_3, D_6)] \oplus V(\omega_6, D_6).$$

Notice that because of the highest-lowest weight duality the top components $v_2^{(1)}$ and $v_4^{(6)}$ can be arranged in two differentials of another self-dual complex. By analogy with [59] we have

Conjecture 6.6. *The open set U_{Gor} is equal to the set U_{split} of points where the complex $\mathbb{F}_\bullet^{top}$ is split exact.*

Here we will prove a partial result

Theorem 6.7. *We have $U_{split} \subset U_{Gor}$.*

Proof. We apply the technique of reverse calculation by starting with the split exact complex, and calculating higher structure theorems $v_2^{(1)}$ and $v_4^{(6)}$ for this complex with the generic defect variables. Then we prove that the resulting self-dual complex of length 4 over a polynomial ring $S := \mathbb{C}[Z_7]$ on the defect variables is a resolution of a cyclic S-module S/J_7 where J_7 is a Gorenstein ideal of codimension 4 with 7 generators. $\square$

Let us perform the computation in the split exact case. We start with the split exact complex, with the basis of F being $\{f_1, \ldots, f_7\}$, basis of G being $\{\bar{e}_1, \ldots, \bar{e}_6, e_6, \ldots, e_1\}$. The matrices of the differentials are

$$d_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$d_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$d_2(e_i) = f_i (1 \le i \le 6), d_2(e_7) = 0, d_2(\bar{e}_i) = 0, 1 \le i \le 7,$$

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The map $\tilde{a}_3 : R \to V(\omega_5, D_6)$ sends $1 \mapsto u_\emptyset$. So $(\tilde{a}_3)_\emptyset = 1$, $(\tilde{a}_3)_I = 0$ for $I \neq \emptyset$. Since the cycle q_1 is given by the formula

$$u_I \mapsto \sum_{i=1}^{6} (\tilde{a}_3)_{I \cup \{i\}} e_i + \sum_{i=1}^{6} (\tilde{a}_3)_{I \setminus \{i\}} \bar{e}_i$$

we get that $q_1 : V(\omega_6, D_6) \to G$ sends $u_{\{i\}}$ to $\bar{e}_i$ for $1 \le i \le 6$ and sends u_I to 0 for $|I| > 1$. This means that

$$p_1(u_{\{i\}}) = f_i + b_{\{i\}} f_7$$

for $1 \le i \le 6$ and

$$p_1(u_I) = b_I f_7$$

for $|I| > 1$.

Next we calculate the map $v_1^{(1)}$. We use the relations in degree $(1,1)$ with respect to W_1 and W_6. We deal with the representation $F \otimes F^* \otimes G \otimes V(\omega_6, D_6)$. The representations we see in the ring $A(7)_\infty$ are $S_{1,0,0,0,0,0,-1} F \otimes V(\omega_1 + \omega_6, D_6)$ and $\mathbb{C} \otimes V(\omega_5, D_6)$. So the relations are the representations $\mathbb{C} \otimes V(\omega_1 + \omega_6, D_6)$ and $S_{1,0,0,0,0,0,-1} F \otimes V(\omega_5, D_6)$. The entries off diagona l $f_i \otimes f*_j$ for $i \neq j$ give relations between $d_4 v_1^{(1)}$ and $d_3 p_1$. The entries $f_i \otimes f_i^*$ on diagonal give also additional 3 term relations involving $d_4 v_1^{(1)}$, $d_3 p_1$ and $\tilde{a}_3$. Making the calculation we see that $v_1^{(1)}$ is a tensor

$$v_1^{(1)} = \sum_{i=1}^{6} \sum_{|I| even, i \notin I} b_{I \cup \{i\}} f_i \otimes u_I + f_7 \otimes u_\emptyset.$$

The calculation of the top complex in this case was recently completed using the programs of Xianglong Ni [41]. The defect variables are b_{ijk} and b_{ijklm} where lower indices are from $[1,6]$ and the variabes are skew-symmetric in lower indices. There is also an additional variable c. We set the degree of variables b to be equal to 1 and degree of c to be 2. The matrix d_2 has the form

$$\begin{pmatrix} U_{1,1} & U_{2,1} & U_{3,1} & U_{4,1} & U_{5,1} & U_{6,1} & U_{7,1} \\ U_{1,2} & U_{2,2} & U_{3,2} & U_{4,2} & U_{5,2} & U_{6,2} & U_{7,2} \\ U_{1,3} & U_{2,3} & U_{3,3} & U_{4,3} & U_{5,3} & U_{6,3} & U_{7,3} \\ U_{1,4} & U_{2,4} & U_{3,4} & U_{4,4} & U_{5,4} & U_{6,4} & U_{7,4} \\ U_{1,5} & U_{2,5} & U_{3,5} & U_{4,5} & U_{5,5} & U_{6,5} & U_{7,5} \\ U_{1,6} & U_{2,6} & U_{3,6} & U_{4,6} & U_{5,6} & U_{6,6} & U_{7,6} \\ U_{1,\bar{1}} & U_{2,\bar{1}} & U_{3,\bar{1}} & U_{4,\bar{1}} & U_{5,\bar{1}} & U_{6,\bar{1}} & U_{7,\bar{1}} \\ U_{1,\bar{2}} & U_{2,\bar{2}} & U_{3,\bar{2}} & U_{4,\bar{2}} & U_{5,\bar{2}} & U_{6,\bar{2}} & U_{7,\bar{2}} \\ U_{1,\bar{3}} & U_{2,\bar{3}} & U_{3,\bar{3}} & U_{4,\bar{3}} & U_{5,\bar{3}} & U_{6,\bar{3}} & U_{7,\bar{3}} \\ U_{1,\bar{4}} & U_{2,\bar{4}} & U_{3,\bar{4}} & U_{4,\bar{4}} & U_{5,\bar{4}} & U_{6,\bar{4}} & U_{7,\bar{4}} \\ U_{1,\bar{5}} & U_{2,\bar{5}} & U_{3,\bar{5}} & U_{4,\bar{5}} & U_{5,\bar{5}} & U_{6,\bar{5}} & U_{7,\bar{5}} \\ U_{1,\bar{6}} & U_{2,\bar{6}} & U_{3,\bar{6}} & U_{4,\bar{6}} & U_{5,\bar{6}} & U_{6,\bar{6}} & U_{7,\bar{6}} \end{pmatrix}$$

Here the upper 6×7 block has the entries $U_{i,j}$ to be skew-symmetric in i,j for $1\le i,j\le 6$ with

$$U_{1,2} = b_{12345}b_{126} - b_{12346}b_{125} + b_{12356}b_{124} - b_{12456}b_{123},$$

The other entries $U_{i,j}$ we get by permuting the indices $1,2,\ldots,6$. and $U_{i,7}=0$ for $1\le i\le 6$. These entires are akin to Cartan relations for pure spinors in D_5.

The entries $U_{i,\bar{j}}$ are given as follows.

$$U_{7,\bar{j}} = b_{\hat{j}},$$

And for $1\le j\le 6$ we have

$$U_{1,\bar{6}} = b_{123}b_{145} - b_{124}b_{135} + b_{125}b_{134},$$

with other entries $U_{i,\bar{j}}$ are gotten by permuting indices $1,2,\ldots,6$. These entries are akin to 4×4 Pfaffians. Finally the entries $U_{i,\bar{i}}$ have 11 entries each, with c and ten entries $b_{ijk}b_{lmn}$ with appropriate signs. The details of this resolution will be published in [42].

6.3.3. *Case* $n=8$. The ring $A(8)_\infty$ decomposes as follows

$$A(8)_\infty = \oplus_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6),a,b} S_{(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6,0,-a)}F\otimes V(\mu(\lambda,a,b)),$$

where

$$\mu(\lambda,a,b) = \begin{array}{ccccccccccc} (\lambda_1-\lambda_2) & - & (\lambda_2-\lambda_3) & - & (\lambda_3-\lambda_4) & - & (\lambda_4-\lambda_5) & - & \lambda_5 & - & (\lambda_6+b) \\ & & & & & & & & | & & \\ & & & & & & & & \lambda_6 & & \\ & & & & & & & & | & & \\ & & & & & & & & a & & \end{array}.$$

Let us look at the decompositions of critical representations. We have

$$W_1 = F\otimes(V(\omega_1,D_7)\oplus V(\omega_6,D_7)\oplus[\underline{so}(14)\oplus\mathbb{C}]\oplus V(\omega_7,D_7)\oplus V(\omega_1,D_7),$$

$$\begin{gathered} W_7 = F^*\otimes(\mathbb{C}\oplus V(\omega_7,D_7)\oplus[V(\omega_1,D_7)\oplus V(\omega_2,D_7)]\oplus[V(\omega_1+\omega_7,D_7)\oplus V(\omega_6,D_7)]\oplus \\ \oplus[V(\omega_3,D_7)\oplus 2*V(\omega_1,D_7)\oplus V(\omega_2,D_7)\oplus\mathbb{C}]\oplus[V(\omega_1+\omega_6,D_7)\oplus \\ \oplus V(\omega_7,D_7)]\oplus[V(\omega_1,D_7)\oplus V(\omega_2,D_7)]\oplus V(\omega_6,D_7)\oplus\mathbb{C}), \end{gathered}$$

The representation W_0 is much bigger. It has 11 graded components

$$W_0 = \oplus_{i=0}^{10} W_0^{(i)}.$$

It is self-dual, which means $W_0^{(i)} = W_0^{(10-i)}$. We have

$$W_0^{(0)} = V(\omega_6, D_7),$$

$$W_0^{(1)} = V(\omega_4, D_7) \oplus V(\omega_2, D_7) \oplus \mathbb{C},$$

$$W_0^{(2)} = V(\omega_2 + \omega_7, D_7) \oplus V(\omega_1 + \omega_6, D_7) \oplus 2^*V(\omega_7, D_7)$$

$$W_0^{(3)} = V(\omega_5, D_7) \oplus V(\omega_1 + \omega_4, D_7) \oplus V(2\omega_7, D_7) \oplus 2^*V(\omega_3, D_7) \oplus V(\omega_1 + \omega_2, D_7) \oplus 2^*V(\omega_1, D_7),$$

$$W_0^{(4)} = V(2\omega_1 + \omega_6, D_7) \oplus V(\omega_2 + \omega_6, D_7) \oplus 3^*V(\omega_1 + \omega_7, D_7) \oplus V(\omega_3 + \omega_7, D_7) \oplus 2^*V(\omega_6, D_7)$$

$$W_0^{(5)} = 2^*V(\omega_4, D_7) \oplus 2^*V(\omega_1 + \omega_3, D_7) \oplus V(\omega_6 + \omega_7, D_7) \oplus V(\omega_1 + \omega_5, D_7) \oplus 2^*V(2\omega_1, D_7) \oplus 2^*V(\omega_2, D_7) \oplus \mathbb{C}$$

Notice that because of the highest-lowest weight duality the top components $v_4^{(1)}$ and $v_8^{(7)}$ can be arranged in two differentials of another self-dual complex $\mathbb{F}_\bullet^{top}$.

The calculation of the top complex in this case was recently completed using the programs of Xianglong Ni [41]. The defect variables are b_{ijk}, b_{ijklm} and $b_{1234567}$ where lower indices are from $[1, 7]$ and the variabes are skew-symmetric in lower indices. There are also seven additional variables c_{ijklmn}. They are also skew-symmetric in lower indices. We set the degree of variables b to be equal to 1 and degree of c to be 2. The matrix d_2 has the form

$$\begin{pmatrix}
U_{1,1} & U_{2,1} & U_{3,1} & U_{4,1} & U_{5,1} & U_{6,1} & U_{7,1} & U_{8,1} \\
U_{1,2} & U_{2,2} & U_{3,2} & U_{4,2} & U_{5,2} & U_{6,2} & U_{7,2} & U_{8,2} \\
U_{1,3} & U_{2,3} & U_{3,3} & U_{4,3} & U_{5,3} & U_{6,3} & U_{7,3} & U_{8,3} \\
U_{1,4} & U_{2,4} & U_{3,4} & U_{4,4} & U_{5,4} & U_{6,4} & U_{7,4} & U_{8,4} \\
U_{1,5} & U_{2,5} & U_{3,5} & U_{4,5} & U_{5,5} & U_{6,5} & U_{7,5} & U_{8,5} \\
U_{1,6} & U_{2,6} & U_{3,6} & U_{4,6} & U_{5,6} & U_{6,6} & U_{7,6} & U_{8,6} \\
U_{1,7} & U_{2,7} & U_{3,7} & U_{4,7} & U_{5,7} & U_{6,7} & U_{7,7} & U_{8,7} \\
U_{1,\bar{1}} & U_{2,\bar{1}} & U_{3,\bar{1}} & U_{4,\bar{1}} & U_{5,\bar{1}} & U_{6,\bar{1}} & U_{7,\bar{1}} & U_{8,\bar{1}} \\
U_{1,\bar{2}} & U_{2,\bar{2}} & U_{3,\bar{2}} & U_{4,\bar{2}} & U_{5,\bar{2}} & U_{6,\bar{2}} & U_{7,\bar{2}} & U_{8,\bar{2}} \\
U_{1,\bar{3}} & U_{2,\bar{3}} & U_{3,\bar{3}} & U_{4,\bar{3}} & U_{5,\bar{3}} & U_{6,\bar{3}} & U_{7,\bar{3}} & U_{8,\bar{3}} \\
U_{1,\bar{4}} & U_{2,\bar{4}} & U_{3,\bar{4}} & U_{4,\bar{4}} & U_{5,\bar{4}} & U_{6,\bar{4}} & U_{7,\bar{4}} & U_{8,\bar{4}} \\
U_{1,\bar{5}} & U_{2,\bar{5}} & U_{3,\bar{5}} & U_{4,\bar{5}} & U_{5,\bar{5}} & U_{6,\bar{5}} & U_{7,\bar{5}} & U_{8,\bar{5}} \\
U_{1,\bar{6}} & U_{2,\bar{6}} & U_{3,\bar{6}} & U_{4,\bar{6}} & U_{5,\bar{6}} & U_{6,\bar{6}} & U_{7,\bar{6}} & U_{8,\bar{6}} \\
U_{1,\bar{7}} & U_{2,\bar{7}} & U_{3,\bar{7}} & U_{4,\bar{7}} & U_{5,\bar{7}} & U_{6,\bar{7}} & U_{7,\bar{7}} & U_{8,\bar{7}}
\end{pmatrix}$$

The entries in this matrix are much more complicated.

In the upper 7×8 block the entries $U_{i,j}$ with $i, j \in [1, 7]$ and with $i \neq j$ have degree 4 and 226 terms each. The entries $U_{i,i}$ have that have degree 4 and 506 terms each. The entries $U_{8,1}$ have degree 3 and 31 terms each.

The entries $U_{i,\bar{j}}$ are have the following properties. The entries $U_{i,\bar{j}}$ for $i, j \in [1, 7]$, $i \neq j$ have degree 4 and 346 terms each. The entries $U_{i,\bar{i}}$ have degree 4 and 756 terms each. The entries $U_{8,\bar{j}}$ have degree 3 and 66 terms each. The details of this description will be published in [42].

One checked by computer [41] that this description agrees with the description given below in the section on Schubert varieties. However for E_8 one can only implement the matrix d_3, but one can still prove the acyclicity of the resulting complex using Buchsbaum-Eisenbud acyclicity criterion.

By analogy with [59] we have

Conjecture 6.8. *The open set U_{Gor} is equal to the set U_{split} of points where the complex $\mathbb{F}_\bullet^{top}$ is split exact.*

Here we will prove a partial result

Theorem 6.9. *We have $U_{split} \subset U_{Gor}$.*

Proof. We apply the technique of reverse calculation by starting with the split exact complex, and calculating higher structure theorems $v_4^{(1)}$ and $v_8^{(7)}$ for this complex with the generic defect variables. Then we prove that the resulting self-dual complex of length 4 over a polynomial ring $S = \mathbb{C}[Z_8]$ on the defect variables is a resolution of a cyclic S-module $S(8)/J_8$ where J_8 is a Gorenstein ideal of codimension 4 with 8 generators. □

6.4. **The affine case** $n = 9$**.** The Lie algebra $\mathbb{L}_\bullet$ is periodic of period 2 with $\mathbb{L}_{2i+1} = V(\omega_8, D_8)$, $\mathbb{L}_{2i} = V(\omega_2, D_8)$. It would be interesting to analyze the "Tom and Jerry" examples of Miles Reid from this point of view, i.e. to calculate the structure theorems p_i for these cases.

6.5. **Connection to opposite Schubert varieties.** The pattern related to E_6, E_7, E_8 triplets seems to extend to the case of general E_n. Consider the homogeneous space $G(E_n)/P_1$ where P_1 is the maximal parabolic related to the first node of E_n (in the convention of Bourbaki, it is the extremal node on the arm of length two). The opposite Schubert varieties in this homogeneous space are indexed by the cosets $W(E_n)/W(D_{n-1})$. This set is in natural bijection with the $W(E_n)$-orbit of the fundamental weight ω_1. Let us look at the poset of the opposite Schubert varieties in low codimension. We draw the case of E_7 but the case of general E_n is completely analogous.

We have one opposite Schubert variety in each codimension up to codimension 3:

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{matrix}$$

$$\begin{matrix} -1 & 1 & 0 & 0 & 0 & 0 \\ & & 0 & & & \end{matrix}$$

$$\begin{matrix} 0 & -1 & 1 & 0 & 0 & 0 \\ & & 0 & & & \end{matrix}$$

$$\begin{matrix} 0 & 0 & -1 & 1 & 0 & 0 \\ & & 1 & & & \end{matrix}$$

Then in codimension 4 we have two:

$$\sigma_4' = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 0 \\ & & -1 & & & \end{matrix}, \sigma_4'' = \begin{matrix} 0 & 0 & 0 & -1 & 1 & 0 \\ & & 1 & & & \end{matrix}$$

The first one is our Gorenstein variety. Its equations are extremal generalized Plücker coordinates: one in each codimension listed above, and

$$\begin{matrix} 0 & 0 & 0 & 0 & -1 & 1 \\ & & 1 & & & \end{matrix}, \quad \begin{matrix} 0 & 0 & 0 & 0 & 0 & -1 \\ & & 1 & & & \end{matrix}$$

In codimensions 5 and 6, 7 in total (for general E_n, $n \geq 6$, we get n defining equations). Let us denote the big opposite cell in $G(E_n)/P_1$ by $Z(n)$ and the codimension 4 opposite Schubert variety which is orenstein by $Z(n)_{\sigma'_4}$

I conjecture that in each of the cases E_6, E_7, E_8 the varieties $Z(n)_{\sigma'_4}$ are Gorenstein and that the ideals J_n we got in the previous sections by reverse calculations give linear sections of these Schubert varieties. If one would have $U_{Gor} = U_{split}$ this would indicate that these Schubert varieties would be the generic Gorenstein ideals of codimension 4 with $6, 7, 8$ generators respectively.

Remark 6.10. *(1) In the paper* [18] *with Sara Filippini and Jacinta Torres we have already shown that the relevant Schubert variety in $(E_6)/P_1$ is the generic hyperplane section in codimension 3 Pfaffians. So if the implication $U_{Gor} = U_{split}$ would be true here would allow to prove the generic form of Gorenstein ideals of codimension 4 with 6 generators (or maybe we could use the existing result of Herzog-Miller since we are clearly in "generically complete intersection" case. The graded format of the resolution is in fact*

$$0 \to R(-6) \to R^5(-4) \oplus R(-5) \to R^{10}(-3) \to R(-1) \oplus R^5(-2) \to R.$$

(2) For type E_7 we expect the resolution of the format

$$0 \to R(-10) \to R^6(-7) \oplus R(-6) \to R^{12}(-5) \to R(-4) \oplus R^6(-3) \to R.$$

The numerator of Hilbert series is

$$1 + 4x + 10x^2 + 14x^3 + 10x^4 + 4x^5 + x^6.$$

(3) For type E_8 we expect the resolution of the format

$$0 \to R(-22) \to R^7(-15) \oplus R(-14) \to R^{14}(-11) \to R(-8) \oplus R^7(-7) \to R.$$

The numerator of Hilbert series is

$$1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + 84x^6 + 113x^7 + 136x^8 + 146x^9 +$$
$$+ 136x^{10} + 113x^{11} + 84x^{12} + 56x^{13} + 35x^{14} + 20x^{15} + 10x^{16} + 4x^{17} + x^{18}.$$

(4) There are two more intriguing Gorenstein Schubert varieties in the homogeneous spaces of E_6, E_7, E_8. One is a codimension 5 one in in (E_7, P_7). We analyzed it in [18] *and it turns out to be a generic Huneke-Ulrich ideal of codimension 5 and deviation 2 (7 generators). The other one is in (E_8, P_8). It is a codimension 6 deviation 2 Gorenstein ideal (8 generators).*

(5) Note, however, that even proving $U_{Gor} = U_{split}$ does not imply that our Schubert varieties are generic, as we need to show that for any specific Gorenstein resolution over some local ring S, and the map $\phi : A(n)_\infty \to S$ the preimage of the maximal ideal in S intersects U_{split}.

There are also explicit descriptions of the hyperplane sections of two Schubert varieties corresponding to E_7 and E_8. One obtains them if one forgets the variables c with the index 2 at the distinguished node

Proposition 6.11. *(1) The case E_7. We consider the ideal in the symmetric algebra of the half-spinor representation $V(\omega_6, D_6)$ generated by the invariant Δ of degree 4 and its partial derivatives with respect to the coordinates which have one negative coordinate. Identifying the variables with b_i, b_{ijk}, b_{ijklm} (the lower indices indicate negative coordinates) we see that our ideal is generated by Δ and $\frac{\partial\Delta}{\partial b_i}$ for $i = 1, \ldots, 6$. Forgetting the variables b_i we get a resolution of the complex $\mathbb{F}_\bullet^{top}$ described above when we actually remove the variable c there. This was checked by Macaulay 2 [41].*

(2) The case E_8. We consider the ideal in the symmetric algebra of the half-spinor representation $V(\omega_7, D_7)$ generated by the invariant Δ of degree 8 and its partial derivatives with respect to the coordinates which have one negative coordinate. Identifying the variables with b_i, b_{ijk}, b_{ijklm} and $b_{1234567}$ (the lower indices indicate negative coordinates) we see that our ideal is generated by Δ and $\frac{\partial\Delta}{\partial b_i}$ for $i = 1, \ldots, 7$. Forgetting the variables b_i we get a resolution of the complex $\mathbb{F}_\bullet^{top}$ described above when we actually remove the variables c_{ijklmn} there. This was checked by Macaulay 2 [41].

Remark 6.12. *In cases E_6, E_7, E_8 one can analyze the degrees of the structure maps for the resolutions of the Schubert varieties mentioned above. Since these resolutions are graded, all stucture maps can be chosen to be homogeneous.*

In all three cases we have that the first graded component

$$\underline{g}_1(E_n) = V(\omega_{n-1}, D_{n-1}).$$

$$W_{n-1} = F^* \otimes [\mathbb{C} \oplus V(\omega_{n-1}, D_{n-1}) \oplus \ldots]$$

so the first two graded components are d_4 and p_1. From this we can figure out the degrees of the elements of $\underline{g}_1(E_n)$. We denote the basis of F by $\{f_1, \ldots, f_n\}$. The last generator is the one of higher degree.

(1) Type E_6. We have $deg(\tilde{a}_3) = 2$ so for the entries of p_1 we have

$$deg(f_i^* \otimes u_I) = 1, 1 \leq i \leq 5,$$

$$deg(f_6^* \otimes u_I) = 0.$$

This gives $deg(u_I) = -1$ for basis elements u_I of $\underline{g}_1(E_6)$. We see that the degrees of top components of W_1 and W_5 are 0 and -1, which suggests an element of U_{split}. Also, the top component of W_0 (whose lowest component is $\tilde{a}_3$) has degree 0.

(2) Type E_7. We have $deg(\tilde{a}_3) = 4$ so for the entries of p_1 we have

$$deg(f_i^* \otimes u_I) = 1, 1 \leq i \leq 6,$$

$$deg(f_7^* \otimes u_I) = 3.$$

This gives $deg(u_I) = -1$ for basis elements u_I of $\underline{g}_1(E_7)$. We see that the degrees of top components of W_1 and W_6 are 0 and -1, which suggests an element of U_{split}. Also, the top component of W_0 (whose lowest component is $\tilde{a}_3$) has degree 0.

(3) Type E_8. We have $deg(\tilde{a}_3) = 10$ so for the entries of p_1 we have

$$deg(f_i^* \otimes u_I) = 6, 1 \leq i \leq 7,$$

$$deg(f_8^* \otimes u_I) = 7.$$

This gives $deg(u_I) = -1$ for basis elements u_I of $\underline{g}_1(E_8)$. We see that the degrees of top components of W_1 and W_7 are 0 and -1, which suggests an element of U_{split}. Also, the top component of W_0 (whose lowest component is $\tilde{a}_3$) has degree 0.

(4) Type E_9. We have resolutions of format

$$0 \to R(-6) \to R^9(-4) \to R^{16}(-3) \to R^9(-2) \to R$$

For these resolutions we have $deg(\tilde{a}_3) = 3$ so for the entries of all structure maps p_i we have

$$deg(f_i^* \otimes u_I) = 2, 1 \leq i \leq 9.$$

This gives $deg(u_I) = 0$ for basis elements u_I of $\underline{g}_1(E_9)$. So in each of representations W_1, W_8, W_0 each homogeneous component has the same degree (1 in W_1, 2 in W_8 and 3 in W_0). This is the same behavior as for the "tame" formats for resolutions of length 3.

(5) It seems plausible that for $n \geq 9$ the Gorenstein ideal of codimension 4 being licci is related to the property that the structure maps p_i become zero for $i >> 0$. For $n \leq 8$ we expect all of them to be licci.

6.6. **Quadratic relations for general n.** This section contains s some remarks about general n. We look at the quadratic relations among generators of $A(n)_\infty$ from W_1, W_{n-1}, W_0. Their interpretation gives new insight into our structure.

We have

$$\begin{aligned} W_1 &= F \otimes [G \oplus V(\omega_{n-2}, D_{n-1}) \oplus \ldots], \\ W_{n-1} &= F^* \otimes [\mathbb{C} \oplus V(\omega_{n-1}, D_{n-1}) \oplus ...] \\ W_0 &= \mathbb{C} \otimes [V(\omega_{n-1}, D_{n-1}) \oplus ...]. \end{aligned}$$

We have

$$\begin{aligned} v_0^{(1)} &= d_3, \\ v_0^{(n-1)} &= d_4, v_1^{(n-1)} = p_1, \\ v_0^{(0)} &= \tilde{a}_3. \end{aligned}$$

Let us analyze the relations of degree $(1,1,0)$. They contain the relations of types

$$\mathbb{C} \otimes ..., S_{1,0^{n-2},-1}F \otimes ...$$

the first type of relations has the first graded component $d_3 d_4 = 0$, and in the next degree we see the three terms relation with terms $d_4 v_1^{(1)}$, $d_3 p_1$ and $\tilde{a}_3$.

This means that in general the graded components of W_0 are quadratic polynomials in graded components of W_1 and W_{n-1}. This is why above, apart from $\tilde{a}_3$, we do not indicate what they are.

The relations of type $S_{1,0^{n-2},-1}F \otimes ...$ have only two terms $d_4 v_1^{(1)}$ and $d_3 p_1$ and closer analysis reveals they imply that the spinor coordinates $(\tilde{a}_3)_I$ are all linear combinations of the entries of d_4, i.e. spinor coordinates are in the resolved ideal. Thus the interpretation of the components of $v_1^{(1)}$ is that they are coefficients of components of $\tilde{a}_3$ when written as linear combinations of the generators of the resolved ideal.

This means we get the following application.

Theorem 6.13. *(1) Let R be a normal local ring, and let $\mathbb{F}_\bullet$ will be a minimal free resolution of the R-module R/J where J is a Gorenstein ideal of codimension 4. Assume that $\mathbb{F}_\bullet$ has a multiplicative structure and that $\mathbb{F}_\bullet$ has a spinor structure map $\tilde{a}_3$. Then the spinor coordinates are in J. In particular all of the above is true if R is complete of characteristic $\neq 2$.*

(2) If R is a polynomial ring over a field K of characteristic $\neq 2$ and J is a homogeneous Gorenstein ideal of codimension 4 then the assumptions above are satisifed and the same conclusion holds.

Remark 6.14. *(1) The tensors corresponding to structure maps p_i are contained in the representation W_{n-1}.*

Let us look at the quadratic relations of degree $(0, 2, 0)$.

7. A deformation of Gorenstein ideals of codimension 4 given by complexes $\mathbb{F}_\bullet^{toptop}$

Let us first state the conjecture.

Conjecture 7.1. *(LICCI Conjecture) Every Gorenstein ideal of codimension 4 with ≤ 8 generators is licci.*

Here is what is needed to be proved to establish this conjecture.

Theorem 7.2. *We have the following two facts.*

(1) The reverse calculation applied to the split exact complex gives the complex $\mathbb{F}_\bullet^{top}$ which is a resolution of the Gorenstein Schubert variety of codimension 4 of type E_6, E_7, E_8.

(2) The defining ideal I of the Gorenstein Schubert variety of codimension 4 of type E_6, E_7, E_8 is licci.

Proof. These facts are established using Macaulay 2. □

Let $S = K[X_1, \ldots, X_m]$ be a polynomial ring over a field K of characteristic 0. Let J be a Gorenstein ideal of codimension 4 over S such that S/J has a finite free resolution

$$\mathbb{G}_\bullet : 0 \to S \to F \otimes S \to G \otimes S \to F^* \otimes S \to S$$

of the format $(1, n, 2n-2, n, 1)$. Let us denote two sets of defect variables for the format $(1, n, 2n-2, n, 1)$ by $\{b_I\}$ and $\{b'_I\}$.

We also denote $T' = S[\{b'_I\}]$ and let $\mathbb{H}_\bullet$ be a complex we get by calculating the top complex for the split complex $\mathbb{H}_\bullet$ of format $(1, n, 2n-2, n, 1)$, using the set of variables $\{b'_I\}$.

We have

Proposition 7.3. *(1) The pairs $(S, \mathbb{G}_\bullet$ and $(T', \mathbb{F}_\bullet)$ are generalized localizations of each other in the sense of Huneke-Ulrich (*[29], [52]*).*

(2) If the defining ideal of a complex $\mathbb{H}^{top}$ of a split exact complex $\mathbb{H}_\bullet$ of format $(1n, 2n-2, n, 1)$ is licci, then the ideal J resolved by $\mathbb{G}_\bullet$ is licci.

Proof. Consider the polynomial ring $T = S[\{b_I\}, \{b'_I\}]$ and over ring T consider the complex $\tilde{\mathbb{G}}_\bullet$ with the differentials given by generic lifts of all HST's given by the generic ring $A(n)_\infty$

in that case. Note that the complex $\tilde{\mathbb{G}}_\bullet$ is acyclic by Peskine-Szpiro Acyclicity Lemma, as in order to prove its acyclicity it is enough to do it for localizations T_P where P are prime ideals with $depth_{T_P} P \le 3$. But for such prime ideals the localization of the ideal J is a unit ideal. So after this localization we deal essentially with the resolution of our Schubert variety which we know is acyclic. Note that the same argument shows that all ideals J, J^{top} and $(J^{top})^{top}$ over S, T and T' have the corresponding cyclic modules with resolutions of format $(1, f_1, f_2, f_3)$.

We have a diagram

$$\begin{tikzcd} & (T, \tilde{\mathbb{G}}_\bullet) \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ (S, \mathbb{G}_\bullet) & & (T', \mathbb{F}_\bullet) \end{tikzcd}$$

We need to define maps ϕ and ψ and show that they are both complete intersections. The mapr ϕ is just dividing T by a regular sequence (b_I, b'_I) where b_I, b'_I are two sets of defect variables. The resulting ideal is just the original ideal J. The map ψ is dividing by the ideal $(X_i - x_i^{(0)}, b_I - b_i^{(0)}))$ where $(x_i^{(0)}, b_I^{(0)})$ are coordinates of a particular point in $SpecS[\{b_I\}]$ which is not in the zero set of the ideal J^{top}. This proves the first part of the proposition.

The ideal J^{top} involves only variables X_i and b_I, so we can think of it as the ideal in T', But there its resolution is split, so the complex $\mathbb{F} \otimes_T T'$ is the top complex of a split exact complex. Therefore it is the resolution of our Schubert variety, i.e. the corresponding ideal is licci. Then we apply Lemma 1.11 from [52]. This proves the proposition.

□

Remark 7.4. *The hope was that the above results would establish the LICCI Conjecutre. The idea was to try to use the results of Huneke-Ulrich, in particular Lemma 1.1 from* [52] *to see that these ideals are essential deformations of each other. Unfortunately the second deformation specializing the variables X_i is not local, so one cannot use the Lemma.*

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