NOTES ON FINITE FREE RESOLUTIONS

JERZY WEYMAN

Abstract. These notes on free resolutions of length 3 (and beyond) incorporate the developments related to the ICERM workshop from August 2020. They constitute an introduction to the recent progress on finite free resolutions resulting from the ideas of the paper [49], and connection to Schubert varieties described in [41]. The notes start with three general sections on Kac-Moody Lie algebras. Then, in order to make it more accessible to commutative algebraists one finds some gentler sections on representation theory. The notes continue with the basic commutative algebra results on acyclicity of complexes and the structure of finite free resolutions. One recalls the Buchsbaum-Eisenbud multipliers theory and proceed to the case of resolutions of length 3. The proof of the results of [49] is included. Then the notes proceed to some examples of resolutions of small formats. Finally one looks at the structure of generic ring and go through the Dynkin types to give the complete description of their spectra. The notes finish with describing connection with Schubert varieties and applications to perfect ideals of codimension 3.

1. Introduction.

In the paper [49] I constructed specific generic rings $\hat{R}_{gen}$ for finite free resolutions of length three of all formats. The structure of these generic rings is related to T-shaped graphs $T_{p,q,r}$ where $(p,q,r) = (r_1+1, r_2-1, r_3+1)$ where (r_1, r_2, r_3) are the ranks of differentials in our complex. I also gave sets of generators for these rings. The main consequence of the construction was that the generic ring $\hat{R}_{gen}$ is Noetherian if and only if the associated graph $T_{p,q,r}$ is Dynkin. Thus we can talk of Dynkin formats and we expect the structure of resolutions to be particularly simple for Dynkin formats.

The results of [49] lead to specific conjectures about the open subset of the spectrum $Spec(\hat{R}_{gen})$ where the generic complex resolves perfect module. In particular one can construct very interesting resolutions of perfect ideals of codimension 3 which are defining ideals of certain affine parts of Schubert varieties in the appropriate homogeneous spaces. They follow definite pattern which extends beyond Dynkin formats.

These notes give an account of the ideas related to these results. I tried to put together the results of the old paper [47] and of [49] to get a unified approach.

The structure of the notes is as follows. W start with the explanation of the representation theory notions we need. The next four section give the basic material from commutative algebra. Next we give a definition of a generic ring, the account of Hochster's solution of the case $n = 2$. Then we proceed with the proof in the case $n = 3$.

We postpone the results on the representation theory of Kac-Moody Lie algebra of the graph $T_{p,q,r}$ to the appendix.

The remaining sections contain the examples for Dynkin formats. We give explicit description of the generic rings for these formats, describe the Zariski open sets U_{CM} where the generic resolution os a resolution of a perfect module.

Finally we give examples from algebraic geometry (Artinian algebras, points in $\mathbb{P}^2$, $\mathbb{P}^3$, curves in $\mathbb{P}^3$, $\mathbb{P}^4$, surfaces in $\mathbb{P}^4$, $\mathbb{P}^5$) of the occurrence of resolutions of Dynkin formats.

1.1. **Acknowledgements.** I discussed the issues related to these notes with many persons.

First, I would like to thank David Buchsbaum and David Eisenbud who introduced me to this subject. I benefitted over the years from discussions with Kaan Akin, Ela Celikbas, Lars Christensen, Harm Derksen, Sara Filippini, Lorenzo Guerrieri, Craig Huneke, Joachim Jelisiejew, Witold Kraśkiewicz, Andrew Kustin, Joe Landsberg, Jai Laxmi, Andras Lorincz, Claudia Miller, Claudia Polini, Piotr Pragacz, Claudiu Raicu, Steven Sam, Jacinta Torres, Bernd Ulrich, Oana Veliche.

There were several events dedicated to resolutions of length 3 in recent years. They were all important for the development of the ideas of these notes. In August 2019 there was a summer school at UCSD organized by Steven Sam. In August 2020 there was an ICERM conference on resolutions of length 3. In the Fall of 2020 there was a remotely run seminar on resolutions of length 3 at MSRI. These notes were written and updated during that time. I would like to thank the participants of all these events.

2. Generalities on Kac-Moody Lie algebras.

1. Kac-Moody Lie algebras

In this section we recall the basic definitions of Kac-Moody Lie algebras. The main reference is [29].

Let A be *a generalized Cartan matrix*, i.e. an $n \times n$ integer matrix

$$A = (a_{i,j})_{1 \le i,j \le n}$$

such that $a_{i,i} = 2$ for $i = 1, \dots, n$, $a_{i,j} \le 0$ for $i \ne j$ and such that $a_{i,j} = 0$ implies $a_{j,i} = 0$.

We will actually assume that A is *symmetrizable*, i.e. there exist a diagonal matrix D with diagonal entries $\epsilon_1, \dots, \epsilon_n$ and a symmetric matrix B such that

$$A = DB.$$

Let $l = rank(A)$. Consider the complex vector space $\mathfrak{h}$ of dimension $2n - l$. We take $\Pi = \{\alpha_1, \dots, \alpha_n\}$ in $\mathfrak{h}^*$ to be the coordinate functions. This is the basis of *simple roots*. We take $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ in $\mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}.$$

Thus we can think of $\alpha_i^\vee$ as the i-th column of A. The set $\Pi^\vee$ is a basis of *simple coroots*.

The *Kac-Moody Lie algebra* $\mathfrak{g}(A)$ is the Lie algebra generated by elements e_i, f_i, $1 \le i \le n$, with the following defining relations:

$$[h, h'] = 0 \ for \ h, h' \in \mathfrak{h},$$
$$[h, e_j] = \langle h, \alpha_j \rangle e_j, [h, f_j] = -\langle h, \alpha_j \rangle f_j,$$
$$[e_i, f_j] = \delta_{i,j} \alpha_i^\vee,$$
$$ad(e_i)^{1-a_{i,j}}(e_j)) = ad(f_i)^{1-a_{i,j}}(f_j) = 0 \ for \ i \ne j.$$

Let $Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i$, $Q_+ = \oplus_{i=1}^n \mathbb{Z}_+\alpha_i$, and $Q_- = -Q_+$. We define a partial ordering $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ has *the root space decomposition* $\mathfrak{g} = \oplus_{\alpha \in Q}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x\}$. An element $\alpha \in Q$ is called *a root* if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. The number $mult(\alpha) = dim\ \mathfrak{g}_\alpha$ is called *the multiplicity* of the root α. A root $\alpha > 0$ (resp. $\alpha < 0$) is called *positive* (resp. *negative*). One can easily show that every root is either positive or negative. We denote by $\Delta, \Delta_+, \Delta_-$ the sets of roots, positive and negative roots respectively.

For $\alpha = \sum_{i=1}^n k_i\alpha_i \in Q$ the number $ht(\alpha) := \sum_{i=1}^n k_i$ is called *the height* of α. We define *the principal gradation* on $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ by setting $\mathfrak{g}_j = \oplus_{ht(\alpha)=j}\mathfrak{g}_\alpha$. Note that $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_{-1} = \oplus\mathbb{C}f_j$, $\mathfrak{g}_1 = \oplus\mathbb{C}e_i$. Let $\mathfrak{g}_\pm = \oplus_{j\geq 1}\mathfrak{g}_{\pm j}$. Then we have *the principal triangular decomposition*

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-.$$

The Weyl group of A is a subgroup of $Aut(\mathfrak{h}^*)$ generated by *the simple reflections*

$$r_i(\lambda) = \lambda - \langle\lambda, \alpha_i^\vee\rangle\alpha_i.$$

for $\lambda \in \mathfrak{h}^*$.

We choose the weight $\rho \in \mathfrak{h}^*$ by requiring

$$\langle\rho, \alpha_i^\vee\rangle = 1,\ \ for\ i = 1, \ldots, n.$$

A $\mathfrak{g}$-module V is $\mathfrak{h}$-*diagonalizable* if $V = \oplus_{\lambda\in\mathfrak{h}^*}V_\lambda$ where $V_\lambda = \{v \in V \mid h{\cdot}v = \lambda(h)v\ \forall h \in \mathfrak{h}\}$ is the λ *weight space*. If $V_\lambda \neq 0$ then λ is a weight of V. The number $mult_\lambda(V) := dim\ V_\lambda$ is called *the multiplicity* of λ in V. When all the weight spaces are finite dimensional we define *the character* of V to be

$$ch\ V = \sum_{\lambda\in\mathfrak{h}^*}(dimV_\lambda)e^\lambda,$$

where e^λ are the basis elements of the group algebra $\mathbb{C}[\mathfrak{h}^*]$ with the binary operation $e^\lambda e^\mu = e^{\lambda+\mu}$.

Let $P(V)$ be the set of weights in V and let $D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}$. We define the category $\mathcal{O}$ as follows: its objects are $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules with finite dimensional weight spaces such that there exist finitely many elements $\mu_1, \ldots, \mu_s$ with $P(V) \subset \cup_{i=1}^s D(\mu_i)$, and the morphisms are $\mathfrak{g}$-module homomorphisms. An $\mathfrak{h}$-diagonalizable module V is said to be integrable if all the e_i, f_i $(i = 1, \ldots, n)$ are locally nilpotent on V. All the integrable modules in the category $\mathcal{O}$ are completely reducible ([29], Corollary 10.7).

A $\mathfrak{g}$-module V is called *a highest weight module* with highest weight λ if there is a nonzero vector $v \in V$ such that (i) $\mathfrak{g}_+ v = 0$, (ii) $h{\cdot}v = \lambda(h)v$ for all $h \in \mathfrak{h}$, (iii) $U(\mathfrak{g}){\cdot}v = V$. The vector v is called *a highest weight vector*. Let $\mathfrak{b}_+ = \mathfrak{h} + \mathfrak{g}_+$ be the Borel subalgebra of $\mathfrak{g}$ and $\mathbb{C}_\lambda$ the one dimensional $\mathfrak{b}$-module defined by $\mathfrak{g}_+ 1 = 0$, $h{\cdot}1 = \lambda(h)1$ for $h \in \mathfrak{h}$. The induced module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ is called *the Verma module* with highest weight λ. Every highest weight $\mathfrak{g}$-module with highest weight λ is a quotient of $M(\lambda)$. The Verma module contains a unique maximal proper submodule $J(\lambda)$. Hence the quotient $V(\lambda) := M(\lambda)/J(\lambda)$ is irreducible, and we have a bijection between $\mathfrak{h}^*$ and the set of irreducible modules in the category $\mathcal{O}$ given by $\lambda \mapsto V(\lambda)$.

If λ is *dominant integral* i.e. $\lambda(\alpha_i^\vee)\in\mathbb{Z}_+$ for all $i=1,\dots,n$, then $V(\lambda)$ is integrable and we have the Weyl-Kac character formula ([29], Theorem 10.4)

$$ch\ V(\lambda)=\frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{dim\ \mathfrak{g}_\alpha}}.$$

Here ρ is given by $\rho(\alpha_i^\vee)=1$ for $i=1,\dots,n$. When $\lambda=0$ we obtain the *denominator identity*

$$\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}=\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{dim\ \mathfrak{g}_\alpha}$$

2. Kostant formula.

Let us choose a subset $S\subset\Pi$. This defines the grading on the Kac-Moody Lie algebra

$$\mathfrak{g}(A)=\oplus_{m\in\mathbb{Z}}\mathfrak{g}(A)_m^{(S)}.$$

For $m\neq 0$ the component $\mathfrak{g}(A)_m^{(S)}$ is the span of root spaces $\mathfrak{g}(A)_\alpha$ where α is a root which written in a basis of simple roots has m as a sum of coefficients of α_i with $\alpha_i\notin S$. Such m is denoted $ht^S(\alpha)$. For $m=0$ $\mathfrak{g}(A)_0^{(S)}$ also includes the Cartan subalgebra $\mathfrak{h}$. We denote

$$\mathfrak{g}(A)_+^{(S)}=\oplus_{m>0}\mathfrak{g}(A)_m^{(S)},\mathfrak{g}(A)_-^{S)}=\oplus_{m<0}\mathfrak{g}(A)_m^{(S)},$$

so we have

$$\mathfrak{g}(A)=\mathfrak{g}(A)_+^{(S)}\oplus\mathfrak{g}(A)_0^{(S)}\oplus\mathfrak{g}(A)_-^{(S)}.$$

We also define the subalgebra $\mathfrak{g}_S$, Weyl group W_S, Δ_S, $\Delta_S^\vee$ as the objects defined for the Cartan matrix $A'=(a_{k,l})_{k,l\in S}$. So W_S is generated by reflections r_k, $k\in S$.

Define $\Delta(S)^\pm=\Delta^\pm\setminus\Delta_S^\pm$ and similarly for $\Delta(S)$.

We also define the subset

$$W(S)=\{w\in W\mid \Phi_w\subset\Delta_+(S)\}.$$

where $\Phi_w=\{\alpha\in\Delta^+\mid w^{-1}(\alpha)<0\}$.

Let $\mathbb{C}$ be a trivial $\mathfrak{g}$-module. The homology modules $H_k(\mathfrak{g}_-^{(S)},\mathbb{C})$ are obtained from the complex of $\mathfrak{g}_0^{(S)}$-modules

$$\dots\bigwedge^k(\mathfrak{g}_-^{(S)})\xrightarrow{d_k}\bigwedge^{k-1}(\mathfrak{g}_-^{(S)})\to\dots\to\bigwedge^1(\mathfrak{g}_-^{(S)})\xrightarrow{d_1}\bigwedge^0(\mathfrak{g}_-^{(S)})\xrightarrow{d_0}\mathbb{C}\to 0,$$

where the differential $d_k:\bigwedge^k(\mathfrak{g}_-^{(S)})\to\bigwedge^{k-1}(\mathfrak{g}_-^{(S)})$ is defined as follows

$$d_k(x_1\wedge\dots\wedge x_k)=\sum_{s<t}(-1)^{s+t}[x_s,x_t]\wedge x_1\dots\wedge\hat{x}_s\wedge\dots\wedge\hat{x}_t\wedge\dots\wedge x_k$$

for $k\geq 2$, $x_i\in\mathfrak{g}_-^{(S)}$, and $d_1=d_0=0$.

For simplicity we write $H_k(\mathfrak{g}_-^{(S)})$ instead of $H_k(\mathfrak{g}_-^{(S)},\mathbb{C})$. Each of the terms $\bigwedge^k(\mathfrak{g}_-^{(S)})$ has a $\mathbb{Z}$-grading induced by that on $\mathfrak{g}_-^{(S)}$. For $j\geq 0$ we define $\bigwedge^k(\mathfrak{g}_-^{(S)})_{-j}$ to be the subspace of $\bigwedge^k(\mathfrak{g}_-^{(S)})$

spanned by the vectors of the form $x_1 \wedge \ldots \wedge x_k$ such that $deg(x_1) + \ldots + deg(x_k) = -j$. The homology module $H_k(\mathfrak{g}_-^{(S)})$ also has the induced $\mathbb{Z}$-grading. Note that $\bigwedge^k(\mathfrak{g}_-^{(S)})_{-j} = H_k(\mathfrak{g}_-^{(S)})_{-j} = 0$ for $k > j$. The $\mathfrak{g}_0^{(S)}$-module structure of the homology modules $H_k(\mathfrak{g}_-^{(S)})$ is determined by the following formula known as *Kostant's formula*.

Theorem 2.1. *(*[19], [36]*)*

$$H_k(\mathfrak{g}_-^{(S)}) = \oplus_{w \in W(S), l(w)=k} V_S(w\rho - \rho),$$

where $V_S(\lambda)$ denotes the integrable highest weight $\mathfrak{g}_0^{(S)}$-module with highest weight λ.

3. Parabolic version of a BGG resolution.

Definition 2.2. *Let $S \subset \{1, \ldots, n\}$. Let $\mathfrak{p}^{(S)} = \oplus_{j \geq 0}(\mathfrak{g}^{(S)})_{-j}$ be the parabolic subalgebra. Define a generalized Verma module*

$$M(\lambda)^{(S)} = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^{(S)})} L_S(\lambda)$$

where $V_S(\lambda)$ is considered as a $\mathfrak{g}_0^{(S)} \oplus \mathfrak{g}_+^{(S)}$-module where $\mathfrak{g}_+^{(S)}$ acts trivially.

Theorem 2.3. *(*[33]*, section 9.2) There exists an exact complex of $\mathfrak{p}^{(S)}$-modules*

$$\ldots \to F^p_{(S)} \to \ldots \to F^1_{(S)} \to F^0_{(S)} \to V(\lambda) \to 0$$

where

$$F^p_{(S)} = \oplus_{w \in W'_{(S)}, l(w)=p} M(w^{-1} \cdot \lambda)^{(S)}.$$

Here $V(\lambda)$ is the irreducible $\mathfrak{g}$-module of highest weight λ, $w \cdot \lambda := w(\lambda + \rho) - \rho$, and

$$W'_{(S)} = \{w \in W \mid l(wv) \geq l(w)\ \forall v \in W_{(S)}\}.$$

where $W_{(S)}$ denotes the subgroup of W generated by r_i ($i \in S$). Thus $W'_{(S)}$ is the set of elements of minimal length in the cosets of $W_{(S)}$ (there is one in each coset).

3. The Kac-Moody Lie algebra of type $T_{p,q,r}$

We will be interested in the special diagrams $T_{p,q,r}$ defined as follows

$$
\begin{array}{ccccccccccccc}
x_{p-1} & - & x_{p-2} & \dots & x_1 & - & u & - & y_1 & \dots & y_{q-2} & - & y_{q-1} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \dots & & & & & & \\
 & & & & & & z_{r-2} & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r-1} & & & & & &
\end{array}
$$

We recall some basic notions about Kac-Moody Lie algebra $\mathfrak{g}(T_{p.q.r})$ associated to this diagram. The generalized Cartan matrix $A(T_{p,q,r})$ has rows and columns indexed by the set $\{0, 1, \dots, p-1, 1', \dots, (q-1)', 1'', \dots, (r-1)''\}$ corresponding to the vertices $u, x_1, \dots, x_{p-1}, y_1, \dots, y_{q-1}, z_1, . .$ respectively. Sometimes we denote vertices by natural numbers from $[1, p+q+r-2]$, in the order listed above.

The entries of A are given by

$$
A(T_{p,q,r})_{i,j} = \begin{cases} 2 & if\ i = j; \\ -1 & if\ the\ nodes\ i\ and\ j\ are\ incident\ in\ T_{p,q,r}; \\ 0 & otherwise. \end{cases}
$$

We set $n := p + q + r - 2$ so $A(T_{p,q,r})$ is an $n \times n$ matrix. The following is an easy consequence of results in [29]

Proposition 3.1.
a) If $T_{p,q,r}$ is a Dynkin diagram, then the matrix $A(T_{p,q,r})$ has rank n. The quadratic form corresponding to $A(T_{p,q,r})$ is positive definite,
b) If $T_{p,q,r}$ is an affine Dynkin diagram, then the matrix $A(T_{p,q,r})$ has rank $n-1$. The quadratic form corresponding to $A(T_{p,q,r})$ is semi-positive definite,
a) In all other cases the matrix $A(T_{p,q,r})$ has rank n. The quadratic form corresponding to $A(T_{p,q,r})$ has signature $(n-1, 1)$,

Proof. The first two statements are special cases of Theorem 4.3, the last is exercise 4.6 from [29]. □

Let us describe the roots, coroots and the Weyl group. We take the vector space $\mathfrak{h}$ of dimension n if $T_{p,q,r}$ is not affine and $n+1$ if it is.

Assume first that $T_{p,q,r}$ is not affine. We take $\Pi = \{\alpha_1, \dots, \alpha_n\}$ in $\mathfrak{h}^*$ to be the coordinate functions. This is the basis of simple roots. We take $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ in $\mathfrak{h}$ such that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a_{i,j}.$$

Thus we can think of $\alpha_i^\vee$ as the i-th column of $A(T_{p,q,r})$. The set $\Pi^\vee$ is a basis of simple coroots.

As defined in the previous section, the Weyl group of A is a subgroup of $Aut(\mathfrak{h}^*)$ generated by the simple reflections.

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i. \tag{1}$$

for $\lambda \in \mathfrak{h}^*$. For the graph $T_{p,q,r}$ (or any tree) there is also a combinatorial formula

$$r_i(\lambda_1, \dots, \lambda_n) = (\lambda_1, \dots, \lambda_n) - 2\lambda_i\alpha_i + \sum_{x-i} \lambda_i\alpha_x. \tag{2}$$

This means that to calculate the value of the reflection r_i on λ (thought of as a graph $T_{p,q,r}$ with labeled vertices), we reverse the sign of a label at i, and add this label to the labels of all neighbors of i.

For the affine $T_{p,q,r}$ the above formulas are still true (one has to remember that $\alpha_1^\vee, \dots, \alpha_n^\vee$ are still independent because they also have a component on the $(n+1)$'st coordinate α_{n+1} which is not a part of a basis of simple roots).

We now specialize to $A = A(T_{p,q,r})$ and to $S = [1,n]\setminus\{p+q\}$. This means the distinguished root is the root corresponding to the vertex z_1. We will write $\mathfrak{g} := \mathfrak{g}(T_{p,q,r})$ and $\mathfrak{g}_i := \mathfrak{g}_i^{(S)}$ in this case. We have the following proposition.

Proposition 3.2. *We have*

- *a)* $\mathfrak{g}_0 = \mathfrak{sl}_{r-1} \times \mathfrak{sl}_{p+q} \times \mathbb{C}$,
- *b)* $\mathfrak{g}_1 = \mathbb{C}^{r-1} \otimes \bigwedge^p \mathbb{C}^{p+q}$,
- *c)*
$$\mathfrak{g}_2 = \bigwedge^2 \mathbb{C}^{r-1} \otimes Ker(S_2(\bigwedge^p \mathbb{C}^{p+q}) \to S_{2^p}\mathbb{C}^{p+q})\oplus$$
$$\oplus S_2\mathbb{C}^{r-1} \otimes Ker(\bigwedge^2(\bigwedge^p \mathbb{C}^{p+q}) \to S_{2^{p-1},1^2}\mathbb{C}^{p+q}).$$
- *d) The higher components $\mathbb{L}_m$ can be defined as cokernels of the graded components of the Koszul complex*
$$(\bigwedge^3 \mathbb{L})_m \to (\bigwedge^2 \mathbb{L})_m \to \mathbb{L}_m \to 0.$$

Proof. We use the generalized Kostant formula to identify $\mathfrak{g}(T_{p,q,r})_{>0}^{(S)}$ for $S = [1,n]\setminus\{p+q\}$. We denote by s_i the simple reflection corresponding to the vertex i, where vertices are labeled by $0, 1, \dots, p-1, 1', \dots, (q-1)', 1'', \dots, (r-1)''$ as in section 3. The only elements of length two in the subgroup $W(S)$ are the elements $s_{1''}s_0$ and $s_{1''}s_{2''}$. We identify a weight with a labeled Dynkin diagram, each vertex labeled by a coefficient of the coresponding fundamental weight. Calculating the corresponding values of $w\rho - \rho$ and using the formula (2) we get

$$s_{1''}s_0\rho - \rho = \begin{array}{ccccccccccccc} 0 & - & 0 & \dots & 1 & - & 0 & - & 1 & \dots & 0 & - & 0 \\ & & & & & & | & & & & & & \\ & & & & & & -3 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & 2 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & \dots & & & & & & \\ & & & & & & 0 & & & & & & \\ & & & & & & | & & & & & & \\ & & & & & & 0 & & & & & & \end{array}$$

with values zero at all other vertices. This (discarding labeling of the vertex $1''$) corresponds to the representation $S_2\mathbb{C}^{r-1}\otimes S_{2^{p-1},1^2}\mathbb{C}^{p+q}$.

$$s_{1''}s_{2''}\rho-\rho=\begin{array}{ccccccccccccc}
0&-&0&\dots&0&-&2&-&0&\dots&0&-&0\\
&&&&&&|&&&&&&\\
&&&&&&-3&&&&&&\\
&&&&&&|&&&&&&\\
&&&&&&0&&&&&&\\
&&&&&&|&&&&&&\\
&&&&&&1&&&&&&\\
&&&&&&|&&&&&&\\
&&&&&&\dots&&&&&&\\
&&&&&&0&&&&&&\\
&&&&&&|&&&&&&\\
&&&&&&0&&&&&&
\end{array}$$

with values zero at all other vertices. This (discarding labeling of the vertex $1''$) corresponds to the representation $\bigwedge^2\mathbb{C}^{r-1}\otimes S_{2^p}\mathbb{C}^{p+q}$. Since these are the only weights in $H_2(\mathfrak{g}_+^{(S)})$, description d) follows. $\square$

Taking $E=\mathbb{C}^{r-1}$, $F=\mathbb{C}^{p+q}$ we denote $\mathbb{L}(p,E,F)$ the positive part

$$\mathbb{L}(p,E,F)=\oplus_{i>0}\mathfrak{g}_i.$$

Proposition 3.3. *The algebra $\mathbb{L}(p,E,F)$ is finite dimensional if and only if one of the following cases occurs.*

- *a) $p=q=2$, $r\geq 2$ arbitrary,*
- *b) $q=r=2$, $p\geq 3$ arbitrary,*
- *c) $q=2$, $r=3$, $p=3,4,5$,*
- *d) $q=3$, $r=2$, $p=3,4,5$,*
- *e) $q=2$, $p=3$, $r=4,5$.*

Proof. Indeed, the listed cases are exactly the cases when $T_{p,q,r}$ is a Dynkin diagram. To be more precise, we have $T_{p,q,r}=D_{r+2}$ in case a), $T_{p,q,r}=D_{p+2}$ in case b), $T_{p,q,r}=E_{p+3}$ in case c), $T_{p,q,r}=E_{p+3}$ in case d) and $T_{p,q,r}=E_{r+3}$ in case e). In the listed cases the algebra $\mathbb{L}(p,E,F)$ is obviously finite dimensional. In the other cases, the positive part of the Kac-Moody Lie algebra is infinite dimensional. $\square$

Let us also calculate the beginning part of the parabolic BGG resolutions for the case under consideration, i.e. $\mathfrak{g}:=\mathfrak{g}(T_{p,q,r})$, $S=\{[1,n]\setminus\{p+q\}\}$.

Proposition 3.4. *Let $\mathfrak{g}:=\mathfrak{g}(T_{p,q,r})$, $S=\{[1,n]\setminus\{p+q\}\}$. Let us consider the highest weight $\lambda=(\lambda_1,\dots,\lambda_n)$. The three initial terms of the parabolic BGG complex described in*

Theorem 2.3 are.

$$\begin{array}{c} M(r_{p+q}r_1\cdot\lambda)^{(S)} \oplus M(r_{p+q}r_{p+q+1}\cdot\lambda)^{(S)} \\ \downarrow \\ M(r_{p+q}\cdot\lambda)^{(S)} \\ \downarrow \\ M(\lambda)^{(S)} \end{array}$$

where $M(\mu)^{(S)}$ denotes the parabolic Verma module.

To make things explicit we identify the weights

$$r_{p+q}\cdot\lambda = (\lambda_1 + \lambda_{p+q} + 1, \ldots, -\lambda_{p+q} - 2, \lambda_{p+q} + \lambda_{p+q+1} + 1, \ldots),$$

where listed components are at vertices $1, p+q, p+q+1$ and not listed ones are the same as in λ.

$$r_{p+q}r_1\cdot\lambda =$$
$$(\lambda_{p+q} - 1, \lambda_1 + \lambda_2 + 1, \ldots, \lambda_1 + \lambda_{p+1} + 1, \ldots, -\lambda_1 - \lambda_{p+q} - 3, \lambda_1 + \lambda_{p+q} + \lambda_{p+q+1} + 1, \ldots)$$

where listed components are at vertices $1, 2, p, p+q, p+q+1$ and not listed ones are the same as in λ.

$$r_{p+q}r_{p+q+1}\cdot\lambda =$$
$$(\lambda_1 + \lambda_{p+q} + \lambda_{p+q+1} + 1, \ldots, -\lambda_{p+q} - \lambda_{p+q+1} - 5, \lambda_{p+q} - 1, \lambda_{p+q+1} + \lambda_{p+q+2} + 1, \ldots)$$

where listed components are at vertices $1, p+q, p+q+1, p+q+2$ and not listed ones are the same as in λ.

4. Representation Theory.

In what follows we will use the representations of general linear groups and a little bit of representation theory of orthogonal Lie algebras. As the main reference we can use [48], chapter 2, however we will use slightly different notation.

A sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_s)$ is a partition of m if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s$ and $|\lambda| := \lambda_1 + \ldots + \lambda_s = m$. We identify the partitions $(\lambda_1, \ldots, \lambda_s)$ and $(\lambda_1, \ldots, \lambda_s, 0)$

Let F be an n-dimensional free module over a commutative ring R. We will denote $S_{(\lambda_1,\ldots,\lambda_n)}F$ the Schur module corresponding to the highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Here $\lambda_1, \ldots, \lambda_n \in \mathbf{Z}$ and $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n)$. Again we define $|\lambda| := \lambda_1 + \ldots + \lambda_n$.

We have

$$S_{(\lambda_1+1,\lambda_2+1,\ldots,\lambda_n+1)}F = S_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}F \otimes \bigwedge^n F,$$

and

$$S_{(\lambda_1,\lambda_2\ldots,\lambda_n)}F^* = S_{(-\lambda_n,-\lambda_{n-1},\ldots,-\lambda_1)}F.$$

If $\lambda_n \geq 0$ the Schur functor can be constructed from tensor powers of F by the usual construction with Young idempotent.

Since we mostly will deal with characteristic zero, let us now assume that the ring R is a $\mathbf{Q}$-algebra. The tensor product $S_\lambda F \otimes S_\mu F$ has a decomposition

$$S_\lambda F \otimes S_\mu F = \oplus_\nu S_\nu F^{\otimes c^\nu_{\lambda,\mu}}$$

where $c^\nu_{\lambda,\mu}$ are the Littlewood-Richardson coefficients. They are non-zero only if $|\lambda| + |\mu| = |\nu|$ and have a combinatorial interpretation given for example in [48], chapter 2.

We also have Cauchy formulas

$$S_t(F\otimes G)=\oplus_{|\lambda|=t,\lambda_n\ge 0}S_\lambda F\otimes S_\lambda G,$$

$$\bigwedge^t(F\otimes G)=\oplus_{|\lambda|=t,\lambda_n\ge 0}S_\lambda F\otimes S_{\lambda'}G$$

Now we turn to representation theory of orthogonal Lie algebra $\underline{so}(2n)$. Let V be an orthogonal space of dimension $2n$. The orthogonal form is denoted $\langle -,-\rangle$. We assume the form is given in the hyperbolic form, i.e. we have a basis $\{v_1,\ldots,v_n,\bar v_1,\ldots,\bar v_n\}$ of V such that $\langle v_i,\bar v_j\rangle=\delta_{i,j}$, $\langle v_i,v_j\rangle=0$, $\langle\bar v_i,\bar v_j\rangle=0$. The maximal toral subalgebra $\underline{h}$ is generated by elements h_i $(1\le i\le n)$, where $h_i(v_i)=v_i$, $h_i(\bar v_i)=-\bar v_i$ and $h_i(v_j)=h_i(\bar v_j)=0$ for $i\ne j$.

The irreducible representations $S_{[\lambda]}V$ correspond to highest weights $\lambda=\sum_{i=1}^n a_i\omega_i$ where $\omega_i=(1^i,0^{n-i})$ for $1\le i\le n-2$, $\omega_{n-1}=((\frac12)^n)$, $\omega_n=((\frac12)^{n-1},-\frac12)$. We write $\lambda=(\lambda_1,\ldots,\lambda_n)$. We have two cases: if λ is a partition (i.e. $a_{n-1}+a_n$ is even) then $S_{[\lambda]}V$ can be constructed from the Schur functor $S_\lambda V$ as an $\underline{so}(2n)$-submodule generated by the canonical tableau (see [18], chapter 18).

The remaining representations have to be constructed using half-spinor representations. These are two representations $S_{[\omega_{n-1}]}V$ and $S_{[\omega_n]}V$ of dimension 2^{n-1} corresponding to highest weights ω_{n-1} and ω_n whose weights are $(\pm\frac12,\ldots,\pm\frac12)$ with number of minus signs even for $S_{[\omega_{n-1}]}V$ and number of minus signs odd for $S_{[\omega_n]}V$.

If we write $V=H\oplus\bar H$ where H is a maximal isotropic space spanned by $v_1,\ldots,v_n$, then

$$S_{[\omega_{n-1}]}V=\oplus_{j\ even}\bigwedge^j H,\quad S_{[\omega_n]}V=\oplus_{j\ odd}\bigwedge^j H.$$

For more information, see [18], chapter 21.

Next we restate this information in terms of Dynkin diagrams. It will be important later when we extend this combinatorics to all diagrams $T_{p,q,r}$. We work here over the field of complex numbers $\mathbb{C}$.

For type A_n the corresponding simple simply connected algebraic group is $SL_{n+1}(\mathbb{C})$. The corresponding Lie algebra is $\underline{sl}(n+1,\mathbb{C})$. The Dynkin graph is

$$x_1\ -\ x_2\ \ldots\ x_{n-1}\ -\ x_n$$

The fundamental representations of $SL_{n+1}(\mathbb{C})$ are the exterior powers $\bigwedge^i(\mathbb{C}^{n+1})$ and they correspond naturally to the node x_i. Their highest weights are the fundamental weights ω_i. They also correspond naturally to the nodes x_i. If we write a weight as a linear combination of fundamental weights we can think of it as labeling of Dynkin diagram by numbers. In this language, which we will use in the notes, the weight ρ (half of the sum of positive roots) is just labeling of each node with 1.

The Weyl group W is the symmetric group S_{n+1} and it is generated by reflections $s_i=(i,i+1)$ which also correspond to the nodes x_i. The Weyl group acts naturally on weights. This action has the following interpretation. The action of the reflection s_i on a weight $(a_1,a_2,\ldots,a_n)$ changes it to $(a_1,\ldots,a_{i-2},a_{i-1}+a_i,-a_i,a_{i+1}+a_i,a_{i+2},\ldots,a_n)$. So we can think of this action as flipping the labeling of the node x_i and adding labeling a_i to neighboring nodes. This way of writing the Weyl group action is nice because it generalizes to all graphs $T_{p,q,r}$. If we write everything in terms of roots, this gives us the usual action of just permuting coordinates. This was the convention used for example in [48].

We also have a dotted Weyl group action

$$w^{\cdot}(\lambda) = w(\lambda + \rho) - \rho$$

which is important in stating Bott theorem.

Let us look at the diagram D_n. The corresponding simply connected simple algebraic group is the $Spin(2n)$ group: a double cover of $SO(2n, \mathbb{C})$. The Lie algebra is $\underline{so}(2n, \mathbb{C})$. The Dynkin diagram is

$$\begin{array}{ccccccccc} x_1 & - & x_2 & \dots & x_{n-3} & - & x_{n-2} & - & x_{n-1} \\ & & & & & & | & & \\ & & & & & & x_n & & \end{array}$$

Let us denote $\mathbb{C}^{2n}$ the vector representation of $\underline{so}(2n, \mathbb{C})$. The fundamental representations are $\bigwedge^i(\mathbb{C}^{2n})$ for $1 \leq i \leq n-2$, and two half-spinor representations $V(\omega_{n-1})$, $V(\omega_n)$ of dimensions 2^{n-1}. Their weights in terms of weights of maximal torus in $SO(2n, \mathbb{C})$ are half-integers and their highest weights are $\omega_{n-1} = ((\frac{1}{2})^n)$ and $\omega_n = ((\frac{1}{2})^{n-1}, \frac{-1}{2})$. These representations are not representations of $SO(2n, \mathbb{C})$, they are representations of $Spin(2n, \mathbb{C})$. Regarding the higher exterior powers of $\mathbb{C}^{2n}$, we have $\bigwedge^{2n-i}(\mathbb{C}^{2n}) = \bigwedge^i(\mathbb{C}^{2n})$ because representation $\mathbb{C}^{2n}$ is self-dual, and the highest weight of $\bigwedge^{n-1} \mathbb{C}^{2n}$ is $\omega_{n-1} + \omega_n$,. representation $\bigwedge^{n-1} \mathbb{C}^{2n}$ is irreducible, representation $\bigwedge^n \mathbb{C}^{2n}$ decomposes to $V(2\omega_{n-1}) \oplus V(2\omega_n)$.

The action of the Weyl group on weights satisfies the same pattern as for type A_n. The action of the reflection s_i on a weight $(a_1, a_2, \dots, a_n)$ flips the labeling of the node x_i and adding labeling a_i to neighboring nodes.

5. Some relevant information about Lie algebras of type E_6, E_7, E_8.

In this section we give some information about Lie algebras of type E_6, E_7, E_8. We will not describe these Lie algebras, some information on that will be given in the next section. We describe the fundamental representations corresponding to extremal nodes, as these will be important in the future.

5.1. $\underline{g}(E_6)$. The Dynkin diagram is (labeling of the nodes is traditional, following Bourbaki).

$$\begin{array}{ccccccccc} x_1 & - & x_3 & - & x_4 & - & x_5 & - & x_6 \\ & & & & | & & & & \\ & & & & x_2 & & & & \end{array}$$

The important fundamental representations are representations $V(\omega_1)$ and $V(\omega_6)$ of dimension 27, they are dual to each other. The representation $V(\omega_2)$ has dimension 78 and it is the adjoint $\underline{g}(E_6)$ itself.

5.2. $\underline{g}(E_7)$. The Dynkin diagram is (labeling of the nodes is traditional, following Bourbaki).

$$\begin{array}{ccccccccccc} x_1 & - & x_3 & - & x_4 & - & x_5 & - & x_6 & - & x_7 \\ & & & & | & & & & & & \\ & & & & x_2 & & & & & & \end{array}$$

The important fundamental representations are representations $V(\omega_1)$-the adjoint $\underline{g}(E_7)$ of dimension 133, and $V(\omega_7)$ of dimension 56. The representation $V(\omega_2)$ has dimension 912.

5.3. $\underline{g}(E_8)$**.** The Dynkin diagram is (labeling of the nodes is traditional, following Bourbaki).

$$\begin{array}{ccccccccccccc} x_1 & - & x_3 & - & x_4 & - & x_5 & - & x_6 & - & x_7 & - & x_8 \\ & & & & | & & & & & & & & \\ & & & & x_2 & & & & & & & & \end{array}$$

The important fundamental representations are representations $V(\omega_8)$-the adjoint $\underline{g}(E_8)$ of dimension 248. The representation $V(\omega_1)$ has dimension 3875. The representation $V(\omega_2)$ has dimension 147250.

5.4. **Infinite cases.** For bigger diagrams $T_{p,q,r}$ we have infinite dimensional Kac-Moody Lie algebras $\underline{g}(T_{p,q,r})$. They have fundamental representations (highest or lowest weights; we will use lowest weights in these notes), they are all infinite dimensional. There is the corresponding (infinite) Weyl group generated by reflections and its action and dotted action on weights follows the pattern described for finite cases.

6. Gradings on Lie algebras.

The combinatorics relating free resolutions of length 3 and Lie algebras is related to some gradings on Lie algebras related to roots. This is also a way of seeing the exceptional Lie algebras in terms of classical ones, so it will make some of the material more accessible.

The general pattern is as follows. Let us consider some Lie algebra related to the diagram of type $T_{p,q,r}$. Let us pick a simple root α_i in the corresponding root system . We will refer to this as the case $(T_{p,q,r}, \alpha_i)$. We have a node x_i corresponding to this node. Then every (positive or negative) root can be written as an integral linear combination of simple roots. We define $\underline{g}_j(T_{p,q,r})$ to be the span of all roots into which α_i comes with coefficient j. The Cartan algebra $\underline{h}$ is part of $\underline{g}_0(T_{p,q,r})$. This defines the grading

$$\underline{g}(T_{p,q,r}) = \oplus_{j\in\mathbb{Z}}\underline{g}_j(T_{p,q,r}).$$

Note that $\underline{g}_j(T_{p,q,r})$ are modules over the smaller Lie algebra corresponding to the graph $T_{p,q,r}$ with the node x_i removed. associated to the node i. We will look at these gradings and restrictions of some fundamental representations. Note that in this decomposition there is an extra copy of $\mathbb{C}$ in $\underline{g}_0(T_{p,q,r})$ because the Cartan algebra becomes smaller by one dimension. For D_n and E_6, E_7, E_8 cases we are only interested in nodes next to the central vertex because these are relevant gradings for our purposes. To work out these gradings for exceptional Lie algebras one first works out $\underline{g}_1$ by the rule saying that it is the tensor product of fundamental representations of smaller root systems which are connected components of $T_{p,q,r}$ with the node x_i subtracted, corresponding to nodes adjacent to the node x_i. Then from the description of roots given (for example) in the Bourbaki tables, it is easy to describe the higher components of the grading.

We proceed to examples.

6.1. **Type A_n.**

Example 6.1. *Consider the diagram A_n and the simple root α_i. This corresponds to decomposing*

$$\mathbb{C}^{n+1} = \mathbb{C}^i \oplus \mathbb{C}^{n-i+1}$$

and then

$$\underline{sl}(n+1,\mathbb{C}) = \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1$$

with

$$\underline{g}_{-1} = Hom(\mathbb{C}^{n-i+1}, \mathbb{C}^i),$$

$$\underline{g}_0 = \underline{sl}(i,\mathbb{C}) \oplus \underline{sl}(n+1-i,\mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = Hom(\mathbb{C}^i, \mathbb{C}^{n+1-i}),$$

6.2. **Type D_n.**

Example 6.2. *Consider the diagram D_n and the simple root α_n. Our orthogonal space is $\mathbb{C}^{2n} = (\mathbb{C}^n)^* \oplus \mathbb{C}^n$. We get the decomposition*

$$\underline{so}(2n,\mathbb{C}) = \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1,$$

with

$$\underline{g}_{-1} = (\bigwedge^2 \mathbb{C}^n)^*,$$

$$\underline{g}_0 = \underline{sl}(n,\mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \bigwedge^2 \mathbb{C}^n.$$

Grading corresponding to the simple root α_{n-1} is practically the same due to the symmetry in the diagram D_n.

Example 6.3. *Consider the diagram D_n and the simple root α_{n-3}. Our orthogonal space is $\mathbb{C}^{2n} = (\mathbb{C}^{n-3})^* \oplus \bigwedge^2 \mathbb{C}^4 \oplus \mathbb{C}^{n-3}$. We get the decomposition*

$$\underline{so}(2n,\mathbb{C}) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2,$$

with

$$\underline{g}_{-2} = (\bigwedge^4 \mathbb{C}^4 \otimes \bigwedge^2 \mathbb{C}^{n-3})^*,$$

$$\underline{g}_{-1} = (\bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^{n-3})^*,$$

$$\underline{g}_0 = \underline{sl}(4,\mathbb{C}) \oplus \underline{sl}(n-3,\mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \bigwedge^2 \mathbb{C}^4 \otimes \mathbb{C}^{n-3}.$$

$$\underline{g}_1 = \bigwedge^4 \mathbb{C}^4 \otimes \mathbb{C}^{n-3}.$$

6.3. **Diagram E_6.**

Example 6.4. (E_6, α_2)

$$\underline{g}(E_6) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2,$$

$$\underline{g}_{-2} = (\bigwedge^6 \mathbb{C}^6)^*,$$

$$\underline{g}_{-1} = (\bigwedge^3 \mathbb{C}^6)^*,$$

$$\underline{g}_0 = \underline{sl}(6, \mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \bigwedge^3 \mathbb{C}^6,$$

$$\underline{g}_2 = \bigwedge^6 \mathbb{C}^6.$$

We have the equality of dimensions

$$1 + 20 + 36 + 20 + 1 = 78.$$

Example 6.5. (E_6, α_3)

$$\underline{g}(E_6) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2,$$

$$\underline{g}_{-2} = (\bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^5)^*,$$

$$\underline{g}_{-1} = (\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^5)^*,$$

$$\underline{g}_0 = \underline{sl}(2, \mathbb{C}) \oplus \underline{sl}(5, \mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^5,$$

$$\underline{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^5.$$

We have the equality of dimensions

$$5 + 20 + 28 + 20 + 5 = 78.$$

Note that the grading of E_6 corresponding to simple root α_5 is very similar because of symmetry of graph E_6.

6.4. **Diagram E_7.**

Example 6.6. (E_7, α_2)

$$\underline{g}(E_7) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2,$$

$$\underline{g}_{-2} = (\bigwedge^6 \mathbb{C}^7)^*,$$

$$\underline{g}_{-1} = (\bigwedge^3 \mathbb{C}^7)^*,$$

$$\underline{g}_0 = \underline{sl}(7, \mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \bigwedge^3 \mathbb{C}^7,$$

$$\underline{g}_2 = \bigwedge^6 \mathbb{C}^7.$$

We have the equality of dimensions

$$7 + 35 + 49 + 35 + 7 = 133.$$

Example 6.7. (E_7, α_3)

$$\underline{g}(E_7) = \underline{g}_{-3} \oplus \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2 \oplus \underline{g}_3,$$

$$\underline{g}_{-3} = (S_{2,1}\mathbb{C}^2 \otimes \bigwedge^6 \mathbb{C}^6)^*,$$

$$\underline{g}_{-2} = (\bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^6)^*,$$

$$\underline{g}_{-1} = (\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^6)^*,$$

$$\underline{g}_0 = \underline{sl}(2, \mathbb{C}) \oplus \underline{sl}(6, \mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^6,$$

$$\underline{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^6,$$

$$\underline{g}_2 = S_{2,1}\mathbb{C}^2 \otimes \bigwedge^6 \mathbb{C}^6.$$

We have the equality of dimensions

$$2 + 15 + 30 + 39 + 30 + 15 + 2 = 133.$$

Example 6.8. (E_7, α_5)

$$\underline{g}(E_7) = \underline{g}_{-3} \oplus \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2 \oplus \underline{g}_3,$$

$$\underline{g}_{-3} = (\bigwedge^3 \mathbb{C}^3 \otimes S_{2,1^4}\mathbb{C}^5)^*,$$

$$\underline{g}_{-2} = (\bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^4 \mathbb{C}^5)^*,$$

$$\underline{g}_{-1} = (\mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^5)^*,$$
$$\underline{g}_0 = \underline{sl}(3,\mathbb{C}) \oplus \underline{sl}(5,\mathbb{C}) \oplus \mathbb{C},$$
$$\underline{g}_1 = \mathbb{C}^3 \otimes \bigwedge^2 \mathbb{C}^5,$$
$$\underline{g}_2 = \bigwedge^2 \mathbb{C}^3 \otimes \bigwedge^4 \mathbb{C}^5,$$
$$\underline{g}_3 = \bigwedge^3 \mathbb{C}^3 \otimes S_{2,1^4}\mathbb{C}^5.$$

We have the equality of dimensions

$$5 + 15 + 30 + 33 + 30 + 15 + 5 = 133.$$

6.5. **Diagram E_8.**

Example 6.9. (E_8, α_2)

$$\underline{g}(E_8) = \underline{g}_{-3} \oplus \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2 \oplus \underline{g}_3,$$
$$\underline{g}_{-3} = (S_{2,1^7}\mathbb{C}^8)^*,$$
$$\underline{g}_{-2} = (\bigwedge^6 \mathbb{C}^8)^*,$$
$$\underline{g}_{-1} = (\bigwedge^3 \mathbb{C}^8)^*,$$
$$\underline{g}_0 = \underline{sl}(8,\mathbb{C}) \oplus \mathbb{C},$$
$$\underline{g}_1 = \bigwedge^3 \mathbb{C}^8,$$
$$\underline{g}_2 = \bigwedge^6 \mathbb{C}^8,$$
$$\underline{g}_3 = S_{2,1^7}\mathbb{C}^8.$$

We have the equality of dimensions

$$8 + 28 + 56 + 64 + 56 + 28 + 8 = 248.$$

Example 6.10. (E_8, α_3)

$$\underline{g}(E_8) = \underline{g}_{-4} \oplus \underline{g}_{-3} \oplus \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2 \oplus \underline{g}_3 \oplus {}_4,$$
$$\underline{g}_{-4} = (S_{2,2}\mathbb{C}^2 \otimes \ S_{2,1^6}\mathbb{C}^7)^*,$$
$$\underline{g}_{-3} = (S_{2,1}\mathbb{C}^2 \otimes \bigwedge^6 \mathbb{C}^7)^*,$$
$$\underline{g}_{-2} = (\bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^7)^*,$$
$$\underline{g}_{-1} = (\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^7)^*,$$
$$\underline{g}_0 = \underline{sl}(2,\mathbb{C}) \oplus \underline{sl}(7,\mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^7,$$

$$\underline{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^7,$$

$$\underline{g}_3 = S_{2,1}\mathbb{C}^2 \otimes \bigwedge^6 \mathbb{C}^7,$$

$$\underline{g}_4 = S_{2,2}\mathbb{C}^2 \otimes S_{2,1^6}\mathbb{C}^7.$$

We have the equality of dimensions

$$7 + 14 + 35 + 42 + 52 + 42 + 35 + 14 + 7 = 248.$$

Example 6.11. (E_8, α_5)

$$\underline{g}(E_8) = \underline{g}_{-5} \oplus \underline{g}_{-4} \oplus \underline{g}_{-3} \oplus \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2 \oplus \underline{g}_3 \oplus \underline{g}_4 \oplus \underline{g}_5,$$

$$\underline{g}_{-5} = (S_{2,1^3}\mathbb{C}^4 \otimes S_{2^5}\mathbb{C}^5)^*,$$

$$\underline{g}_{-4} = (\bigwedge^4 \mathbb{C}^4 \otimes S_{2^3,1^2}\mathbb{C}^5)^*,$$

$$\underline{g}_{-3} = (\bigwedge^3 \mathbb{C}^4 \otimes S_{2,1^4}\mathbb{C}^5)^*,$$

$$\underline{g}_{-2} = (\bigwedge^2 \mathbb{C}^4 \otimes \bigwedge^4 \mathbb{C}^5)^*,$$

$$\underline{g}_{-1} = (\mathbb{C}^4 \otimes \bigwedge^2 \mathbb{C}^5)^*,$$

$$\underline{g}_0 = \underline{sl}(4, \mathbb{C}) \oplus \underline{sl}(5, \mathbb{C}) \oplus \mathbb{C},$$

$$\underline{g}_1 = \mathbb{C}^4 \otimes \bigwedge^2 \mathbb{C}^5,$$

$$\underline{g}_2 = \bigwedge^2 \mathbb{C}^4 \otimes \bigwedge^4 \mathbb{C}^5,$$

$$\underline{g}_3 = \bigwedge^3 \mathbb{C}^4 \otimes S_{2,1^4}\mathbb{C}^5,$$

$$\underline{g}_4 = \bigwedge^4 \mathbb{C}^4 \otimes S_{2^3,1^2}\mathbb{C}^5,$$

$$\underline{g}_5 = S_{2,1^3}\mathbb{C}^4 \otimes S_{2^5}\mathbb{C}^5.$$

We have the equality of dimensions

$$4 + 10 + 20 + 30 + 40 + 40 + 40 + 30 + 20 + 10 + 4 = 248.$$

7. Gradings in affine cases

Let us assume we deal with affine cases. i.e. when the graph $T_{p,q,r}$ is the extended Dynkin graph. In such case the Lie algebra $\underline{g}(T_{p,q,r})$ is infinite dimensional but its graded components are periodic, so they can be explicitly described.

7.1. **Graph $T_{3,3,3}$.** We label the vertices

$$\begin{array}{ccccccccc} x_1 & - & x_3 & - & x_4 & - & x_5 & - & x_6 \\ & & & & | & & & & \\ & & & & x_2 & & & & \\ & & & & | & & & & \\ & & & & x_0 & & & & \end{array}$$

Because of symmetry it is enough to consider the case $(T_{3,3,3}, \alpha_2)$. We think of it as corresponding to resolutions of format $(2, 6, 6, 2)$.

We have

$$\underline{g}(T_{3,3,3})_0 = \underline{sl}(F_3) \times \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{3,3,3})_1 = F_3^* \otimes \bigwedge^3 F_1$$

$$\underline{g}(T_{3,3,3})_2 = (\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1 \otimes F_1) \oplus (S_2 F_3^* \otimes \bigwedge^6 F_1)$$

and we see that $\underline{g}(T_{3,3,3})_0$ and $\underline{g}(T_{3,3,3})_2$ are isomorphic, up to some determinants. In fact we have periodicity of period 2.

7.2. **Graph $T_{2,4,4}$.** We label vertices

$$\begin{array}{ccccccccccccc} x_0 & - & x_1 & - & x_3 & - & x_4 & - & x_5 & - & x_6 & - & x_7 \\ & & & & & & | & & & & & & \\ & & & & & & x_2 & & & & & & \end{array}$$

We have two cases

Example 7.1. $(T_{2,4,4}, \alpha_2)$ *We think of this grading as related to free resolution of format* $(3, 8, 6, 1)$. *We have*

$$\underline{g}(T_{2,4,4})_0 = \underline{sl}(F_3) \times \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{2,4,4})_1 = F_3^* \otimes \bigwedge^4 F_1$$

$$\underline{g}(T_{2,4,4})_2 = (\bigwedge^2 F_3^* \otimes \bigwedge^7 F_1 \otimes F_1) \oplus (S_2 F_3^* \otimes \bigwedge^8 F_1)$$

and we see that $\underline{g}(T_{2,4,4})_0$ *and* $\underline{g}(T_{2,4,4})_2$ *are isomorphic, up to some determinants. In fact we have periodicity of period* 2.

Example 7.2. $(T_{2,4,4}, \alpha_3)$ *We think of this grading as related to free resolution of format* $(1, 6, 8, 3)$. *We have*

$$\underline{g}(T_{2,4,4})_0 = \underline{sl}(F_3) \times \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{2,4,4})_1 = F_3^* \otimes \bigwedge^2 F_1$$

$$\underline{g}(T_{2,4,4})_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1,$$

$$\underline{g}(T_{2,4,4})_3 = \bigwedge^3 F_3^* \otimes \bigwedge^4 5F_1 \otimes F_1) \oplus (S_{2,1} F_3^* \otimes \bigwedge^6 F_1)$$

and we see that $\underline{g}(T_{2,4,4})_0$ *and* $\underline{g}(T_{2,4,4})_3$ *are isomorphic, up to some determinants. In fact we have periodicity of period* 3.

7.3. **Graph** $T_{2,3,5}$**.** We label vertices

$$\begin{array}{ccccccccccccc} x_1 & - & x_3 & - & x_4 & - & x_5 & - & \dots & - & x_8 & - & x_0 \\ & & & & | & & & & & & & & \\ & & & & x_2 & & & & & & & & \end{array}$$

We have three cases

Example 7.3. $(T_{2,3,5}, \alpha_2)$ *We think of this grading as related to free resolution of format* $(2,9,8,1)$*. We have*

$$\underline{g}(T_{2,3,5})_0 = \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{2,3,5})_1 = F_3^* \otimes \bigwedge^3 F_1$$

$$\underline{g}(T_{2,3,5})_2 = S_2F_3^* \otimes \bigwedge^6 F_1$$

$$\underline{g}(T_{2,3,5})_3 = S_3F_3^* \otimes \bigwedge^8 F_1 \otimes F_1$$

and we see that $\underline{g}(T_{2,3,5})_0$ *and* $\underline{g}(T_{2,3,5})_3$ *are isomorphic, up to some determinants. In fact we have periodicity of period* 3.

Example 7.4. $(T_{2,3,5}, \alpha_3)$ *We think of this grading as related to free resolution of format* $(1,8,9,2)$*. We have*

$$\underline{g}(T_{2,3,5})_0 = \underline{sl}(F_3) \times \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{2,3,5})_1 = F_3^* \otimes \bigwedge^2 F_1$$

$$\underline{g}(T_{2,3,5})_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1,$$

$$\underline{g}(T_{2,3,5})_3 = S_{2,1}F_3^* \otimes \bigwedge^6 F_1,$$

$$\underline{g}(T_{2,3,5})_4 = (S_{2,2}F_3^* \otimes \bigwedge^7 F_1 \otimes F_1) \oplus (S_{3,1}F_3^* \otimes \bigwedge^8 F_1).$$

and we see that $\underline{g}(T_{2,3,5})_0$ *and* $\underline{g}(T_{2,3,5})_4$ *are isomorphic, up to some determinants. In fact we have periodicity of period* 4.

Example 7.5. $(T_{2,3,5}, \alpha_5)$ *We think of this grading as related to free resolution of format* $(1,5,9,5)$*. We have*

$$\underline{g}(T_{2,3,5})_0 = \underline{sl}(F_3) \times \underline{sl}(F_1) \times \mathbb{C}$$

$$\underline{g}(T_{2,3,5})_1 = F_3^* \otimes \bigwedge^2 F_1$$

$$\underline{g}(T_{2,3,5})_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1,$$

$$\underline{g}(T_{2,3,5})_3 = \bigwedge^3 F_3^* \otimes \bigwedge^5 F_1 \otimes F_1,$$

$$\underline{g}(T_{2,3,5})_3 = \bigwedge^4 F_3^* \otimes \bigwedge^5 F_1 \otimes \bigwedge^3 F_1,$$

$$\underline{g}(T_{2,3,5})_3 = (\bigwedge^5 F_3^* \otimes \bigwedge^5 F_1 \otimes \bigwedge^4 F_1 \otimes F_1) \oplus (S_{2,1^3} F_3^* \otimes \bigwedge^5 F_1 \otimes \bigwedge^5 F_1).$$

and we see that $\underline{g}(T_{2,4,4})_0$ and $\underline{g}(T_{2,4,4})_5$ are isomorphic, up to some determinants. In fact we have periodicity of period 5.

8. Buchsbaum-Eisenbud Acyclicity Criterion and Peskine-Szpiro Acyclicity Lemma.

We employ the following notation regarding finite free resolutions. In principle all rings we encounter are Noetherian unless otherwise stated. We will consider the complexes of free modules over a Noetherian ring R. By definition these are sequences of homomorphisms

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \to F_1 \xrightarrow{d_1} F_0$$

where $F_i = R^{f_i}$ and $d_{i-1}d_i = 0$ for $i = 2, \ldots, n$. We say that $\mathbb{F}_\bullet$ is *a finite free resolution* or it is *acyclic* if the only non-zero homology module of $\mathbb{F}_\bullet$ is $H_0(\mathbb{F}_\bullet) = M$.

In this case $\mathbb{F}_\bullet$ is a finite free resolution of the R-module M.

After choosing bases on each free module F_i we can think of d_i as an $f_{i-1} \times f_i$ matrix. We denote by *rank* r_i of the linear map d_i to be the maximal size of non-vanishing minor of d_i. The ideals $I(d_i) := I_{r_i}(d_i)$ generated by all minors of d_i of rank r_i are essential for our approach.

Before we start we need some properties of the ideals $I(d_i)$.

Lemma 8.1. *Let $d : F \to G$ be a map of free R-modules.*

1. *The ideal $I_r(d_i)$ generated by $r \times r$ minors of the matrix of d does not depend on the choice of basis in F and G,*
2. *Assume R is local, d has rank r and that $I(d_i) = R$. Then after change of bases in F and G the matrix of the map d can be brought to the form*

$$\begin{bmatrix} 1_r & 0 \\ 0 & 0 \end{bmatrix}$$

where 1_r is an $r \times r$ identity matrix.

We have two criteria for the acyclicity of $\mathbb{F}_\bullet$.

Theorem 8.2. *(Buchsbaum-Eisenbud, [9]) The complex $\mathbb{F}_\bullet$ is acyclic if and only if the following two conditions hold*

1. *$f_i = r_i + r_{i+1}$ for all $1 \le i \le n$, with the convention that $r_{n+1} = 0$.*
2. *For all $1 \le i \le n$ we have $depth(I(d_i)) \ge i$.*

Remark 8.3. *(1) The assumption that the map d_n is injective is essential. Otherwise we have the examples similar to the following. Take $R = K[X]/(X^2)$. Take the complex with $F_i = R$, $d_i = (X)$. We get an complex which is acyclic but $rank(d_i) + rank(d_{i+1}) > f_i$ for all i.*

(2) This statement is true over non-Noetherian rings with the appropriate definition of the grade. This is explained in the book [37] *of Northcott. He defined the true grade*

$$Grade_R(I, M) = sup_{n\geq 0} grade_{R[x_1,\ldots,x_n]}(I \otimes_R R[x_1, \ldots, x_n], M \otimes_R R[x_1, \ldots, x_n])$$

where grade is defined as the maximal length of a regular sequence on M contained in I, and proved that Buchsbaum-Eisenbud acyclicity criterion holds with this definition of the depth over any ring R. Thus we will use the theory over arbitrary rings and in case of (possibly) non-Noetherian ring, depth will mean the true grade in the above sense.

There is another statement which is almost equivalent.

Theorem 8.4. *(Lemme d'Acyclicite, Peskine-Szpiro*[38]*). Let $\mathbb{F}_\bullet$ be a free complex. Then $\mathbf{F}_\bullet$ is acyclic if and only if $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic at all prime ideals P such that* $\operatorname{depth} PR_P < n$.

Proof. We just need to prove that if $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic for all prime ideals P such that $depth PR_P < n$, then $\mathbf{F}_\bullet$ is acyclic. We prove that for every prime P the complex $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic. We do it by induction on $\operatorname{depth} R_P$. If $\operatorname{depth} R_P < n$ then we are done by assumption. So let us assume that $\operatorname{depth} R_P \geq n$. Then we have a complex $\mathbb{F}_\bullet$ of length $n \leq depth\ \mathfrak{m}$ over a local ring $(S, \mathfrak{m})$ such that for every prime ideal $P \neq \mathfrak{m}$ $\mathbb{F}_\bullet \otimes_S S_P$ is acyclic. The result then follows from the lemma.

Lemma 8.5. *Let S be a commutative Noetherian ring, $I \subset S$ an ideal. Assume that we have a complex*

$$\mathbb{M}_\bullet : 0 \to M_n \to M_{n-1} \to \ldots \to M_1 \to M_0$$

of finitely generated S-modules. Denote by H_k the k-th homology module of the complex $\mathbb{M}_\bullet$. Assume that

(1) $depth(I, M_k) \geq k$ for $1 \leq k \leq n$,
(2) $depth(I, H_k) = 0$ for $1 \leq k \leq n$.

Then the complex $\mathbb{M}_\bullet$ is acyclic.

Apply Lemma 8.5 to the complex $\mathbb{F}$ over S and $I = \mathfrak{m}$. Indeed, $depth(I, F_k) = n \geq k$ for $k = 0, \ldots, n$. Also, for every $P \neq \mathfrak{m}$ $(H_k)_P = 0$ which means the only associated prime of M is $\mathfrak{m}$, so $depth(\mathfrak{m}, H_k) = 0$ for $k = 1, \ldots, n$. It remains to prove the lemma.

Let B_k denote the module of boundaries in degree k, Z_k-the modules of cycles in degree k. Let us take the biggest m such that $B_m \neq Z_m$. We will prove $m = 0$. We make the following claims.

(1) $depth(I, B_m) \geq m + 1 \geq 2$,
(2) $depth(I, M_m) \geq m \geq 1$.

Considering the short exact sequence

$$0 \to B_m \to Z_m \to H_m \to 0$$

and the long exact sequence of Ext's, noting that $depth(I, Z_m) \geq 1$ because $Z_m \subset M_m$, we get

$$\ldots 0 = Ext^0(R/I, Z_m) \to Ext^0(R/I, H_m) \to 0$$

which means that $depth(I, H_m) \geq 1$ - contradiction.

Since the second claim is in the assumption of the lemma, it is enough to prove the first one. We use the exact sequence

$$0 \to M_n \to \ldots \to M_{m+1} \to B_m \to 0,$$

we divide it into short exact sequences and use the long exact sequences of Ext's. □

Now we are ready to prove Theorem 8.2. First we prove that the rank and depth conditions are sufficient. We use Theorem 8.4. So it is enough to show that $\mathbb{F}_\bullet \otimes_R R_P$ is acyclic for $depth\ PR_P < n$. But this implies that $I(d_n)_P$ is a unit ideal. This, by row reduction implies that the complex $\mathbb{F}_\bullet \otimes_R R_P$ can be written as a direct sum of the isomorphism $R_P^{f_n} \to R_P^{f_n}$ and the free complex of length $n-1$ of format $(f_0, \ldots, f_{n-2}, r_{n-1})$ satisfying the conditions of Theorem 8.2. By induction on n we see that this complex is indeed acyclic, so we are done.

Finally we prove that rank and depth conditions are necessary. We use induction on n. First we treat the case $n = 1$. We have a map $d_1 : F_1 \to F_0$ and we need to show that if d_1 is injective then $f_1 \leq f_0$ and $I(d_1)$ contains a non-zerodivisor. We note that if d_i is injective then the f_1-st exterior power of d_1 is also injective. Indeed, we have the commutative diagram

$$\begin{array}{ccc} F_1^{\otimes f_1} & \to & F_0^{\otimes f_1} \\ \uparrow & & \uparrow \\ \bigwedge^{f_1} F_1 & \to & \bigwedge^{f_1} F_0 \end{array}$$

where the horizontal maps are induced by d_1 and vertical are the natural injections. It follows that the map $\bigwedge^{f_1} F_1 \to \bigwedge^{f_1} F_0$ has to be injective which gives the rank and the depth condition for $n = 1$.

To prove that the conditions are necessary for $n > 1$ one needs to take the following steps. One takes the multiplicative subset S of non-zerodivisors and then one knows that natural map $f : R \to S^{-1}R$ is injective. So if $\mathbb{F}_\bullet$ is acyclic, then $\mathbb{F}'_\bullet := \mathbb{F}_\bullet \otimes_R (S^{-1}R)$ is acyclic. We denote $d'_i = S^{-1}d_i$. We have $rank\ d'_i = rank\ d_i$ because the map f is injective and localization commutes with taking exterior powers of maps of free modules. This means $I(d'_i) = S^{-1}I(d_i)$. We know by the case $n = 1$ that $I(d_n)$ contains a non-zero divisor. This means that $I(d'_i) = S^{-1}R$.

Now we have the following lemma

Lemma 8.6. *The module $Coker(d'_n)$ is free over $S^{-1}R$.*

Proof. This follows from the general fact that a projective module of constant rank over a semilocal ring is free (Bourbaki, "Commutative Algebra" Chapter 2, section 5, Proposition 5). So we need to show that $Coker(d'_n)$ is projective of constant rank. Therefore it is enough to show that for every prime ideal P in $S^{-1}R$, over a local ring R_P $Coker(d_n \otimes_R R_P)$ is free of rank $rankF_n - rank(d_n)$. But this is clear by 8.1. □

So we reduced the length of our complex, because we need to prove the result for the complex

$$0 \to Coker(d'_n) \xrightarrow{d''_{n-1}} F'_{n-2} \to \ldots \to F'_1 \to F'_0.$$

Note that $I(d''_{n-1}) = I(d'_{n-1})$, it follows from row reduction. We proceed by descending induction on k, we need to show that $I(d'_k) = S^{-1}R$. But if we take the biggest k such that

$I(d'_k) \neq S^{-1}R$, then all the maps d_l for $l > k$ split so we are reduced to the case of the injective map. Then from the case $n = 1$ we see that $I(d'_k)$ has to contain a non-zerodivisor, but in $S^{-1}R$ every non-zerodivisor is a unit, which gives a contradiction.

Splitting all the maps d'_k shows that the rank conditions $r_i + r_{i+1} = rank F_i$ have to be satisfied.

To check the depth conditions, assume that $\mathbb{F}_\bullet$ is acyclic but $depth\ I(d_k) = l < k$ for some k. Let us take the biggest such k. By the standard commutative algebra argument there exists a prime ideal P, $I(d_k) \subset P$ such that $depth\ PR_P = l$. Let us localize at P. Then for all $m > k$ $I(d_m) \otimes_R R_P = R_P$ (because their depth over R was $\geq m$), so they split. However we have that $I(d_k) \otimes_R R_P \neq R_P$. This means that the projective dimension of $H_0(\mathbb{F}_\bullet) \otimes_R R_P$ is equal to k. But since $depth\ PR_P = l < k$, we get a contradiction with the Auslander-Buchsbaum formula

$$pd_{R_P} M + depth(PR_P, M) = depth PR_P.$$

9. The First and Second Structure Theorems of Buchsbaum and Eisenbud.

Throughout we will employ the following notation. We will deal with finite free resolutions

$$\mathbb{F}_\bullet : 0 \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \ldots \to F_1 \xrightarrow{d_1} F_0$$

with $rank(d_i) = r_i$, $F_i = R^{f_i}$ and we will assume $f_i = r_i + r_{i+1}$ for $1 \leq i \leq n$. We refer to the sequence $(f_0, f_1, \ldots, f_n)$ as to *the format* of the resolution $\mathbb{F}_\bullet$.

We have the following First Structure Theorem of Buchsbaum and Eisenbud ([10], Theorem 3.1).

Theorem 9.1. *We have the unique sequence of maps*

$$a_i : \otimes_{j=i}^{n} (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \to \bigwedge^{r_i} F_{i-1}$$

such that

(1) $a_n : \bigwedge^{r_n} F_n \to \bigwedge^{r_n} F_{n-1}$ is just $\bigwedge^{r_n} d_n$,
(2) We have a commutative diagram

$$\begin{array}{ccc}
\bigwedge^{r_i} F_i & \xrightarrow{\bigwedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\
\downarrow = & & \uparrow\ a_i \\
\bigwedge^{f_i} F_i \otimes \bigwedge^{r_{i+1}} F_i^* & \xrightarrow{\bigwedge^{f_i} F_i \otimes a_{i+1}^*} & \otimes_{j=i}^{n} (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}}
\end{array}$$

Proof. In the paper [10] the theorem is stated in the SL-equivariant form. More precisely, one claims the isomorphisms $\bigwedge^{r_i} F_i \equiv \bigwedge^{r_{i+1}} F_i^*$ and one claims the commutativity of the diagram

$$\begin{array}{ccc}
\bigwedge^{r_i} F_i & \xrightarrow{\bigwedge^{r_i} d_i} & \bigwedge^{r_i} F_{i-1} \\
\downarrow = & & \uparrow\ a_i \\
\bigwedge^{r_{i+1}} F_i^* & \xrightarrow{a_{i+1}^*} & R
\end{array}$$

Since the claim follows from certain factorization, both versions are equivalent (once we choose bases in the modules F_i).

The idea of the proof is very simple. Assume that the map a_i was constructed and $i \geq 2$. We will construct the map a_{i-1}. We consider the map

$$\tilde{d}_i : \bigwedge^{r_i} F_{i-1} \to \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

induced by the differential d_i treated as a map in $d_i : R \to F_{i-1} \otimes F_i^*$.

Then one considers the complex

$$0 \to R \xrightarrow{a_i} \bigwedge^{r_i} F_{i-1} \xrightarrow{\tilde{d}_i} \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

We claim this complex is acyclic. It is clear that the Fitting ideal of the differential on the right is the ideal $I(d_i)$. It is clear that $I(a_i)$ has depth ≥ 2 because $i \geq 2$ and $I(d_i) = I(a_{i+1})I(a_i)$. So one needs to prove that our complex satisfies the rank conditions. But this can be done after localizing, i.e. one can assume $\mathbb{F}_\bullet$ is split exact. In this case it is an exercise for the reader.

We also see that the composition

$$\bigwedge^{r_{i-1}} F_{i-2}^* \xrightarrow{\bigwedge^{r_{i-1}} d_{i-1}^*} \bigwedge^{r_{i-1}} F_{i-1}^* = \bigwedge^{r_i} F_{i-1} \xrightarrow{\tilde{d}_i} \bigwedge^{r_i+1} F_{i-1} \otimes F_i^*$$

is zero. From this it follows the existence of the map a making the diagram

$$\begin{tikzcd}
\bigwedge^{r_{i-1}} F_{i-2}^* \arrow[rr, "\bigwedge^{r_{i-1}} d_{i-1}^*"] \arrow[dr, "a"'] & & \bigwedge^{r_{i-1}} F_{i-1}^* \\
& R \arrow[ur, "a_i"'] &
\end{tikzcd}$$

commute. Now we can take $a_{i-1} = a^*$.

$\square$

The main example which was the motivation for that theorem is the case of $n = 2$, the format $(1, m, m-1)$.

Theorem 9.2. *(Hilbert-Burch) Let R be a Noetherian ring and let us assume we have an acyclic complex*

$$0 \to R^{m-1} \xrightarrow{d_2} R^m \xrightarrow{d_1} R.$$

Let us choose basis in our free modules so we can identify d_2 with the $m \times (m-1)$ matrix

$$d_2 = (y_{i,j})$$

and we can identify d_1 with $1 \times n$ matrix $(x_1, \ldots, x_m)$. The there exists a non-zero divisor $a \in R$ such that

$$x_i = (-1)^i a \Delta_i$$

where Δ_i is the determinant of the matrix d_2 with the i-th row omitted.

Buchsbaum and Eisenbud proved also in [10] the Second Structure Theorem which shows how the submaximal exterior power $\bigwedge^{r_i-1} d_i$ factors through F_i^*.

Theorem 9.3. *Let $i \geq 2$. We have a map $b_i : \otimes_{j=i}^{n} (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \otimes F_i^* \to \bigwedge^{r_i - 1} F_{i-1}$ such that the following diagram commutes*

$$\begin{array}{ccc} \bigwedge^{r_i-1} F_i & \xrightarrow{\bigwedge^{r_i-1} d_i} & \bigwedge^{r_i-1} F_{i-1} \\ \downarrow = & & \uparrow\ b_i \\ \bigwedge^{f_i} F_i \otimes \bigwedge^{r_{i+1}+1} F_i^* & \xrightarrow{\bigwedge^{f_i} F_i \otimes (a_{i+1}^*)'} & \otimes_{j=i}^{n} (\bigwedge^{f_j} F_j)^{\otimes(-1)^{j-i}} \otimes F_i^* \end{array}$$

where $(a_{i+1}^*)'$ is the contraction by a_{i+1}^*.

Proof. We show the following facts:

(1) The complex

$$F_i \xrightarrow{a'_{i+1}} \bigwedge^{r_{i+1}+1} F_i \xrightarrow{d_{i+1}} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

is exact

(2) The composition

$$\bigwedge^{r_i-1} F_{i-1}^* \xrightarrow{\bigwedge^{r_i-1} d_i^*} \bigwedge^{r_i-1} F_i^* = \bigwedge^{r_{i+1}+1} F_i \xrightarrow{d_{i+1}} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

is zero.

These two facts imply the existence of a factorization b

$$\begin{array}{ccccc} \bigwedge^{r_i-1} F_{i-1}^* & & \xrightarrow{\bigwedge^{r_i-1} d_i^*} & & \bigwedge^{r_i-1} F_i^* \\ & b \searrow & & \nearrow a'_{i+1} & \\ & & F_i & & \end{array}$$

and we take $b_i = b^*$.

The first fact follows by continuing the resolution to the left with the rest of the complex $\mathbb{F}_\bullet$, i.e. considering the complex

$$0 \to F_n \to \ldots \to F_{i+1} \to F_i \xrightarrow{a'_{i+1}} \bigwedge^{r_{i+1}+1} F_i \xrightarrow{d_{i+1}} \bigwedge^{r_{i+1}+2} F_i \otimes F_{i+1}^*$$

and applying the Buchsbaum-Eisenbud acyclicity criterion to it. The second fact one can check on a split complex (after localizing at the multiplicative set S of non-zerodivisors in R). $\square$

Remark 9.4. *(1) The first and second structure theorems of Buchsbaum and Eisenbud are true over non-Noetherian rings with true grade replacing depth. This is proved in the book of Northcott* [37].

(2) Bruns proved ([7]) *that the first structure theorem is true for complexes acyclic in codimension one, i.e. the complexes $\mathbb{F}_\bullet$ for which depth$(I(d_i)) \geq 2$ for $i \geq 2$ and depth$(I(d_1)) \geq 1$.*

Remark 9.5. *(1) Let $d : F \to G$ be a map of free R-modules of rank r. The map d induces the map $\bigwedge^r d : R \to \bigwedge^r G \otimes \bigwedge^r F^*$. We also have a map $\tilde{\bigwedge}^r d : G \to \bigwedge^{r+1} G \otimes \bigwedge^r F^*$ induced by $\tilde{\bigwedge}^r d$. Prove tha the composition*

$$F \xrightarrow{d} G \xrightarrow{\tilde{\bigwedge}^r d} \bigwedge^{r+1} G \otimes \bigwedge^{r} F^*$$

is zero, $rank \tilde{\bigwedge}^r d = rank\ G - r$ *and that* $I(d) = I(\tilde{\bigwedge}^r d)$.

(2) *Assume that the complex* $\mathbb{F}_\bullet$ *satisfies the rank conditions of Theorem 8.2 and depth* $I(d_i) \geq i + k$ *for* $i = 1, \ldots, n$. *One can use the previous remark to show that the module* $M = H_0(\mathbb{F}_\bullet)$ *is the* k*-th syzygy.*

10. Linkage.

Let R be a commutative local regular ring. Let I be a perfect ideal of codimension c. We have a finite free resolution

$$\mathbb{G}_\bullet : 0 \to G_c \xrightarrow{d_c} G_{c-1} \to \ldots G_1 \xrightarrow{d_1} R \to R/I \to 0$$

Let $(x_1, \ldots, x_c)$ be a regular sequence in I. We have a map of complexes

$$\alpha : K(x_1, \ldots, x_c) \to \mathbb{G}_\bullet$$

extending the ring homomorphism

$$\alpha_0 : R/(x_1, \ldots, x_c) \to R/I.$$

Theorem 10.1. *The dual of the mapping cone* $C(\alpha)_\bullet$ *is the free resolution of the colon ideal*

$$J := (x_1, \ldots, x_c) : I.$$

Proof. We have an exact sequence

$$0 \to M \to R/(x_1, \ldots, x_c) \to R/I \to 0$$

and the point is that the kernel $M = Ext^c(R/J, R)$, so we really have an exact sequence

$$0 \to Ext^c(R/J, R) \to R/(x_1, \ldots, x_c) \to R/I \to 0$$

Similarly we have an exact sequence

$$0 \to Ext^c(R/I, R) \to R/(x_1, \ldots, x_c) \to R/J \to 0$$

which gives the statement of the theorem.

In order to see it, we pass to the ring $\bar{R} = R/(x_1, \ldots, x_c)$. This is a complete intersection ring, so it is Gorenstein. In this ring we have two ideals $\bar{I} = I/(x_1, \ldots, x_c)$, $\bar{J} = J/(x_1, \ldots, x_c)$ and we see that $\bar{I} = 0 : \bar{J}$, $\bar{J} = 0 : \bar{I}$. This means that

$$\bar{I} = Hom_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}), \bar{J} = Hom_{\bar{R}}(\bar{R}/\bar{I}, \bar{R})$$

This means we have exact sequences

$$0 \to Hom_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}) \to \bar{R} \to \bar{R}/\bar{I} \to 0$$

$$0 \to Hom_{\bar{R}}(\bar{R}/\bar{I}, \bar{R}) \to \bar{R} \to \bar{R}/\bar{J} \to 0.$$

But the long exact sequences of Ext associated to regular sequence $(x_1, \ldots, x_c)$ imply that

$$Hom_{\bar{R}}(\bar{R}/\bar{J}, \bar{R}) = Ext^c_R(R/J, R), Hom_{\bar{R}}(\bar{R}/\bar{I}, \bar{R}) = Ext^c_R(R/I, R).$$

This shows our claim. □

We have the following properties of linkage.

Proposition 10.2. *The relation of linkage satisfies the following*

(1) The ideal J is also perfect of codimension c.
(2) The relation is symmetric, i.e. $(x_1, \ldots, x_c) : J = I$,

We define the equivalence relation of linkage on the perfect ideals of codimension c in R to be the smallest equivalence relation containing the relation $I \equiv J$.

In low codimension the relation of linkage is very useful.

Proposition 10.3. *Let I be a perfect ideal of codimension 2. Then I is linked to a complete intersection.*

Proof. The finite free resolution of R/I is

$$0 \to R^{n-1} \stackrel{d_2}{\to} R^n \stackrel{d_1}{\to} R$$

where $d_1 = (x_1, \ldots, x_n)$, for some n. We can assume without loss of generality that (x_1, x_2) is a regular sequence in I (otherwise we change the basis in R^n). Then the mapping cone looks like

$$\begin{array}{ccccccc} 0 & \to & R^{n-1} & \stackrel{d_2}{\to} & R^n & \stackrel{d_1}{\to} & R \\ & & \uparrow\ \alpha_2 & & \uparrow\ \alpha_1 & & \uparrow\ \alpha_0 \\ 0 & \to & R & \stackrel{\beta_2}{\to} & R^2 & \stackrel{\beta_1}{\to} & R \end{array}$$

where β_2, β_1 are the differentials in the Koszul complex on x_1, x_2. Now, looking at the ranks of the maps modulo the maximal ideal $\mathfrak{m}$ in R we see that $\alpha_0 \otimes R/\mathfrak{m}$ has rank 1 and $\alpha_1 \otimes R/\mathfrak{m}$ has rank 2. This means that the minimal resolution of R/J has the format

$$0 \to R^{n-2} \stackrel{d_2'}{\to} R^{n-1} \stackrel{d_1'}{\to} R.$$

Continuing like that we arrive at a complete intersection. □

For perfect ideals of codimension three we can apply similar construction. Assume that we choose a regular sequence (x_1, x_2, x_3) in I such that $x_1, x_2, x_3)$ are part of a minimal system of generators for I(i.e. their cosets are linearly independent modulo $\mathfrak{m}I$). A minimal resolution

$$0 \to R^m \stackrel{d_3}{\to} R^{m+n-1} \stackrel{d_2}{\to} R^n \stackrel{d_1}{\to} R$$

of R/I will then produce a minimal resolution

$$0 \to R^{n-3} \stackrel{d_3'}{\to} R^{m+n-1} \stackrel{d_2'}{\to} R^{m+3} \stackrel{d_1'}{\to} R$$

of the cyclic module R/J, where J is the linked ideal.

This means if we apply the procedure again (link by a regular sequence which is part of minimal generators of our ideal) we arrive at a resolution of format

$$0 \to R^m \stackrel{d_3''}{\to} R^{m+n-1} \stackrel{d_2''}{\to} R^n \stackrel{d_1''}{\to} R$$

again. Since at each stage we have many choices of regular sequences (x_1, x_2, x_3) we can produce many resolutions of the same format from a given one. This gives hope that resolutions of perfect ideals of codimension three occur in nice families, because from one of them we can produce nice families by linkage. Moreover, if for some reason we end up with the ideal I' such that one of the Koszul relations of the ideal I' is among minimal syzygies, then there is additional cancellation, and we link to an ideal J with a smaller resolution of R/J. Such cases could be then handled by induction.

Notice that this method fails in codimension bigger than three, as the minimalization of the mapping cone will usually have much bigger ranks of modules.

11. Buchsbaum-Rim linkage.

Let us generalize the notion of linkage to modules.

We start with defining a Buchsbaum-Rim complex. Let $\phi : F_1 \to F_0$ be a map of free R-modules, with $rank\ F_0 = m$, $rank\ F_1 = m + c - 1$. Let us assume that the depth of the ideal $I_m(\phi)$ of maximal minors of ϕ is equal to c. We define the Buchsbaum-Rim complex of ϕ of length c with terms

$$(B-R)(\phi)_\bullet : 0 \to D_{c-2}F_0^* \otimes \bigwedge^m F_0^* \otimes \bigwedge^{m+c-1} F_1 \to D_{c-3}F_0^* \otimes \bigwedge^m F_0^* \otimes \bigwedge^{m+c-2} F_1 \to \dots$$

$$\dots \to F_0^* \otimes \bigwedge^m F_0^* \otimes \bigwedge^{m+2} F_1 \to \bigwedge^m F_0^* \otimes \bigwedge^{m+1} F_1 \to F_1 \to F_0.$$

All the maps are induced by ϕ and they are linear except the second one which is induced by maximal minors of ϕ.

One has the following proposition which is an easy application of Buchsbaum-Eisenbud acyclicity criterion.

Proposition 11.1. *Let R be a Noetherian ring and let $\phi : F_1 \to F_0$ be a map of free R-modules, with rank $F_0 = m$, rank $F_1 = m + c - 1$. Assume that depth $I_m(\phi) = c$. Then the complex $(B - R)(\phi)$ is acyclic and it resolves a perfect module of codimension c.*

We have a general position lemma.

Proposition 11.2. *Let R be a Noetherian ring and let $\psi : G_1 \to G_0$ be a linear map. Assume that rank $G_0 = m$, $n :=$ rank $G_1 \geq m + c - 1$. Assume that depth $I(\psi) \geq c$. We think of ψ as an $m \times n$ matrix. Then there exists a choice of basis in G_1 such that after changing the basis the depth of the ideal of maximal minors in every $m \times (m + c - 1)$ submatrix of ψ is $\geq c$.*

Proof. This is proved in the paper by [8] by Bruns (Satz 2 there). □

Let R be a commutative local regular ring. Let M be a perfect module of codimension c. We have a finite free resolution

$$\mathbb{G}_\bullet : 0 \to G_c \xrightarrow{d_c} G_{c-1} \to \dots G_1 \xrightarrow{d_1} G_0 \to M \to 0$$

We have to have $rank\ G_1 \geq rank G_0 + c - 1$, as the depth of the ideal $I(d_1)$ equals to c.

We choose the basis of G_1 such that the ideal of maximal minors of the $m \times (m + c - 1)$ submatrix of d_1 given by the first $m + c - 1$ columns is equal to c. Let us denote the linear map given by this submatrix by $\psi : G'_1 \to G_0$. Then we have a comparison map

$$\alpha : (B - R)(\psi)_\bullet \to \mathbb{G}_\bullet.$$

Let us consider the mapping cone $C(\alpha)_\bullet$. We look at the complex $C(\alpha)^*_\bullet$

Proposition 11.3. *The minimal complex homotopically equivalent to complex $C(\alpha)^*_\bullet$ is a free resolution of a perfect module N of codimension c.*

Proof. Consider the long homology sequence associated to the exact sequence

$$0 \to (B-R)(\psi)_\bullet[-1] \to C(\alpha)_\bullet \to \mathbb{G}_\bullet \to 0$$

We get

$$0 \to H_1(C(\alpha)_\bullet) \to H_0((B-R)(\psi)_\bullet) \to H_0(\mathbb{G}_\bullet) \to 0$$

and that all other homology groups of $C(\alpha)_\bullet$ are zero. This means that $H_1(C(\alpha)_\bullet)$ has homological dimension c. But we can find a regular sequence $(x_1, \ldots, x_c)$ annihilating both modules $H_0((B-R)(\psi)_\bullet)$ and $H_0(\mathbb{G}_\bullet)$. This means it also annihilates $H_1(C(\alpha)_\bullet)$, so this module is perfect. □

Denote $H_0((B-R)(\psi)_\bullet) = N$, $Ext^c(H_1(C(\alpha)_\bullet), R) = M'$ Appying the long homology sequence to the dual of the exact sequence of complexes above we get the exact sequences

$$0 \to Ext^c(M', R) \to N \to M \to 0,$$

$$0 \to Ext^c(M, R) \to Ext^c(N, R) \to M' \to 0.$$

The module M' we constructed we call the first link of M with respect to ψ. The Buchsbaum-Rim linkage is an equivalence relation on perfect modules of codimension c generated by this relation.

Let us assume that we found a regular sequence $\underline{x} = (x_1, \ldots, x_c)$ which annihilates N. Then the sequence $\underline{x}$ annihilates also M and M'. We can use now long exact sequences of Ext's associated to regular sequenc $\underline{x}$ and this leads to isomorphisms

$$Ext^c(M,R) = Hom(M, R/(\underline{x})), Ext^c(M',R) = Hom(M', R/(\underline{x})), Ext^c(N,R) = Hom(N, R/(\underline{x})).$$

So we get the exact sequences of Hom's over $R/(\underline{x})$.

$$0 \to Hom(M', R/(\underline{x})) \to N \to M \to 0$$

and

$$0 \to Hom(M, R/(\underline{x})) \to Hom(N, R/(\underline{x})) \to M' \to 0.$$

Let us specialize to codimension 3. In this case the Buchsbaum-Rim complex has the selfdual form

$$0 \to F_0^* \otimes \bigwedge^m F_0^* \otimes \bigwedge^{m+2} F_1 \to \bigwedge^m F_0^* \otimes \bigwedge^{m+1} F_1 \to F_1 \to F_0$$

so we have $Ext^3(N, R) = N$.

This means we get exact sequences

$$0 \to Ext^3(M', R) \to N \to M \to 0$$

and

$$0 \to Ext^3(M, R) \to N \to M' \to 0.$$

The Buchsbaum-Rim relation is obviously symmetric, as we could recover M from M' and N.

12. Increasing depth: ideal transforms and geometric form of Acyclicity Criterion.

We prove the geometric result on acyclicity of free complexes. It is based on homological algebra and it is essential for our approach.

Theorem 12.1. *Let $X = SpecR$, and let $j : U \to X$ be an open immersion. Let*

$$\mathbb{G} : 0 \to G_n \to G_{n-1} \to \ldots \to G_1 \to G_0$$

be a complex of free R-modules (treated as a complex of sheaves over X) such that $\mathbb{G}\mid_U$ is acyclic. Then $H_n(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$, $H_{n-1}(\mathbb{G} \otimes j_* \mathcal{O}_U) = 0$, and the complex $\mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic if and only if $\mathcal{R}^i j_* \mathcal{O}_U = 0$ for $i = 1, \ldots, n-2$.

Proof. Before we start, let us decompose the complex $\mathbb{G}\mid_U$ to short exact sequences. Denoting $B_i = Im(G_{i+1}\mid_U \to G_i\mid_U)$ we have exact sequences

$$0 \to B_i \to G_i\mid_U \to B_{i-1} \to 0$$

for $i = 2, \ldots, n$ (with $B_n = 0$). and

$$0 \to B_1 \to G_1\mid_U \to G_0\mid_U.$$

This induces long exact sequences

$$0 \to j_* B_i \to G_i \otimes j_* \mathcal{O}_U \to j_* B_{i-1} \to R^1 j_* B_i \ldots$$

as well as an exact sequence

$$0 \to j_* B_1 \to G_1 \otimes j_* \mathcal{O}_U \to G_0 \otimes j_* \mathcal{O}_U.$$

Next we show that vanishing of higher direct images implies acyclicity. Indeed, our vanishing implies that $R^i j_* B_{n-s} = 0$ for $1 \le i \le n - s - 1$. So the above exact sequences have last term zero and we get that $\mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic.

To prove the reverse implication let us proceed by induction on n. For $n = 2$ there is nothing to prove. For $n = 3$ we see from the exact sequences that $H_3(\mathbb{G}) = H_2(\mathbb{G}) = 0$ and $H_1(\mathbb{G}) = Ker(R^1 j_* G_3 \to R^1 j_* G_2)$. We need

Lemma 12.2. *Let M be an R-module. Let $\phi : F \to G$ be a map of free R-modules of finite rank. Denote by $I(\phi)$ the ideal of maximal minors of ϕ. Then $\phi \otimes M$ is a monomorphism if and only if $depth_R(I(\phi), M) \ge 1$.*

Proof. This is a special case of Theorem 2, Appendix B from [37]. □

In our case $R^1 j_* \mathcal{O}_U$ is supported on $X \setminus U$ so $Ker(R^1 j_* G_3 \to R^1 j_* G_2) = 0$ implies $R^1 j_* \mathcal{O}_U = 0$, completing the case $n = 3$. Assume the result is proved for $n - 1$ and the complex $j_* \mathbb{G} = \mathbb{G} \otimes j_* \mathcal{O}_U$ is acyclic. By induction (applied to $\mathbb{G}$ truncated at G_1) we have

$$R^1 j_* \mathcal{O}_U = R^2 j_* \mathcal{O}_U = \ldots = R^{n-3} J_* \mathcal{O}_U = 0.$$

Now our long exact sequences imply that $R^{n-3} j_* B_{n-2} = R^{n-4} j_* B_{n-3} = \ldots = R^1 j_* B_2$. We also have the exact sequence

$$0 \to R^{n-3} j_* B_{n-2} \to R^{n-2} j_* G_n \to R^{n-2} j_* G_{n-1}.$$

Also, from the exact sequences we can deduce that

$$H_1(\mathbb{G} \otimes j_*\mathcal{O}_U) = Ker(R^1 j_* B_2 \to R^1 j_* G_2) = R^1 j_* B_2$$

This means that if $H_1(\mathbb{G} \otimes j_*\mathcal{O}_U) = 0$ then $R^1 j_* B_2 = 0$, so the map $R^{n-2} j_* G_n \to R^{n-2} j_* G_{n-1}$. is a monomorphism, which implies by Lemma 5.2 that $R^{n-2} j_* \mathcal{O}_U = 0$.

□

13. Generic Rings.

We consider the free acyclic complexes $\mathbb{F}_\bullet$ (i.e complexes whose only nonzero homology group is $H_0(\mathbb{F}_\bullet)$) of the form

$$\mathbb{F}_\bullet :\ 0 \to F_n \to F_{n-1} \to \ldots \to F_2 \to F_1 \to F_0$$

over commutative Noetherian rings R, with $rank\ F_i = f_i$ $(0 \le i \le n)$, $rank(d_i) = r_i$ $(1 \le i \le n)$. The tuple $(f_0, f_1, \ldots, f_{n-1}, f_n)$ is *the format* of the complex $\mathbb{F}_\bullet$. We always have $f_i = r_i + r_{i+1}$ $(0 \le i \le n)$.

For the resolutions of such format $(f_0, f_1, \ldots, f_{n-1}, f_n)$ we say that a pair $(R_{gen}, \mathbb{F}_\bullet^{gen})$ where R_{gen} is a commutative ring and $\mathbb{F}_\bullet^{gen}$ is an acyclic free complex over R_{gen} is *a generic resolution* of this format if two conditions are satisfied:

(1) The complex $\mathbb{F}_\bullet^{gen}$ is acyclic over R_{gen},
(2) For every acyclic free complex $\mathbb{G}_\bullet$ over a Noetherian ring S there exists a ring homomorphism $\phi : R_{gen} \to S$ such that

$$\mathbb{G}_\bullet = \mathbb{F}_\bullet^{gen} \otimes_{R_{gen}} S.$$

Of particular interest is whether the ring R_{gen} is Noetherian, because it can be shown quite easily (see for example [6]) that a non-Noetherian (non-unique) generic pair always exists.

Theorem 13.1. *For every format $(f_0, f_1, \ldots, f_n)$ a generic ring for resolutions of this format exists. It is, however, not unique, and in general non-Noetherian.*

Proof. We start with the generic complex of format $(f_0, \ldots, f_n)$. Let R_0 be the coordinate ring of the variety of complexes and let

$$\mathbb{F}_\bullet^{(0)} : 0 \to R_0^{f_n} \xrightarrow{d_n} R_0^{f_{n-1}} \to \ldots \to R_0^{f_1} \xrightarrow{d_1} R_0^{f_0}$$

We define inductively a sequence of Noetherian rings R_m $(m \ge 0)$ and complexes $\mathbb{F}_\bullet^{(m)}$ of format $(f_0, \ldots, f_n)$ over the rings R_m as follows. Assume that the pair $(R_m \mathbb{F}_\bullet^{(m)})$ is constructed. Let $H_i^{(m)}$ denotes the i-th homology module of $\mathbb{F}_\bullet^{(m)}$. It is a finitely generated module, let $\{q_1^{(m),i}, \ldots, q_{N(m,i)}^{(m),i}\}$ be the set of generators of the cycle submodule of $R_m^{f_i}$ $(i = 1, \ldots, n)$. We define the ring R_{m+1} as the R_m-algebra generated by the coordinates of elements $\{p_1^{(m),i}, \ldots, p_{N(m,i)}^{(m),i}\}$ in $R_{m+1}^{f_{i+1}}$ such that $d_i(p_j^{(m),i}) = q_j^{(m),i}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, N(m,i)$. This includes setting $\{q_1^{(m),n}, \ldots, q_{N(m,n)}^{(m),n}\}$ equal to zero. We define the complex $\mathbb{F}_\bullet^{(m+1)} := \mathbb{F}_\bullet^{(m)} \otimes_{R_m} R_{m+1}$. We define the ring $R_{gen} = lim_m R_m$ and $\mathbb{F}_\bullet^{gen} = lim_m \mathbb{F}_\bullet^{(m)}$. We need to show that $H_i(\mathbb{F}_\bullet^{gen}) = 0$ for $i = 1, \ldots, n$. Let us take a cycle z in $R_{gen}^{f_i}$. Each coordinate of z has finitely many terms, so all these terms occur already in some R_m. Thus

z comes from a cycle in $R_m^{f_i}$. This means this cycle is a boundary over R_{m+1}, i.e. it is a boundary over R_{gen}. □

14. The case $n = 2$.

In this section we produce an explicit generic ring for the resolutions of length two. This construction was first done by Hochster [23] with later improvements in [40], [42].

Let us fix a format (f_0, f_1, f_2) with the ranks (r_1, r_2), i.e. $f_2 = r_2, f_1 = r_1 + r_2$. Consider three free $\mathbf{Z}$ modules F_0, F_1, F_2 such that $rank\ F_i = f_i$ $(i = 0, 1, 2)$.

We start with a generic complex, i.e. we take the independent variables $X_{j,i}$ and $Y_{k,j}$ $(1 \le i \le f_2, 1 \le j \le f_1,\ 1 \le k \le f_0)$. Consider two matrices $d_2 = (X_{j,i})$ and $d_1 = (Y_{k,j})$. Construct the ring

$$R_0 = \mathbf{Z}[\{X_{j,i}\}, \{Y_{k,j}\}]_{1\le i\le f_2, 1\le j\le f_1, 1\le k\le f_0}/I(f_0, f_1, f_2)$$

where $I(f_0, f_1, f_2)$ is an ideal generated by relations

$$d_2 d_1 = 0, \bigwedge^{r_1+1} d_1 = 0.$$

Over the ring R_0 we have a "generic complex"

$$\mathbf{F}^0_\bullet : F_2 \otimes_{\mathbf{Z}} R_0 \stackrel{d_2^0}{\to} F_1 \otimes_{\mathbf{Z}} R_0 \stackrel{d_1^0}{\to} F_0 \otimes_{\mathbf{Z}} R_0.$$

The differentials d_2^0 (resp. d_1^0) are the linear maps over R_0 given by matrices $(\bar{X}_{j,i}$, $(\bar{Y}_{k,j})$ over R_0 where $\bar{X}_{j,i}$ (resp. $\bar{Y}_{k,j}$) are the cosets of $X_{j,i}$ (resp. $Y_{k,j}$) in R_0.

Obviously this complex has a universality property with respect to all free complexes of format (f_0, f_1, f_2) over commutative rings.

It turns out that the complex $\mathbf{F}^0_\bullet$ is not acyclic over R_0. The reason is that this complex does not satisfy the first structure theorem of Buchsbaum-Eisenbud. Actually this theorem can be reinterpreted in terms of cycles in the first homology group of this complex.

In terms of depth it means that $depth_{R_0} I_{r_2}(d_2^0) = 1$, $depth_{R_0} I_{r_1}(d_1^0) = 1$. In order to increase depth of $I_{r_2}(d_2^0)$ to two we can take the ideal transform of this ideal. Notice that the First Structure Theorem says that we have a factorization of $r_1 \times r_1$ minors of d_1:

$$M(K|J;Y) = \pm M(J'\ |[1, r_2]; X) a_2(K)$$

where I, J, K are subsets of cardinality r_1 and J' is a complement of J in the set $[1, r_1 + r_2]$, and $M(K|J;Y)$ (resp. $M(J'|[1, r_2]; X)$) are minors of Y (resp. X) on rows from K, columns from J (resp. rows from J', columns from $[1, r_2]$). Notice that this means that each of Buchsbaum-Eisenbud multipliers can be written as a factor of a minor of Y by an arbitrary maximal minor of X, so it is in the ideal transform of $I_{r_2}(d_2)$. This adding the Buchsbaum-Eisenbud multipliers to R_0 is necessary to increase the depth of $I_{r_2}(d_2)$ to 2.

It turns out that the ideal transform of $I_{r_2}(d_2^0)$ is exactly the ring R_a we would get from R_0 by adding to it all Buchsbaum-Eisenbud multipliers and dividing by all relations they would satisfy in all specializations. Over R_a we have a complex $\mathbf{F}^a_\bullet := \mathbf{F}^0_\bullet \otimes_{R_0} R_a$ of format (f_0, f_1, f_2). This construction leads to the following.

Theorem 14.1. *The pair* $(R_a, \mathbf{F}^a_\bullet)$ *is a generic pair for resolutions of length two, of format* (f_0, f_1, f_2). *It has a universality property with respect to all acyclic free complexes of format*

(f_0, f_1, f_2) over commutative rings. The ring R_a has a filtration (which in characteristic zero is a direct sum decomposition)

$$R_a = \oplus_{a,b,\alpha,\beta} S_{(\boldsymbol{a}-\boldsymbol{b}+\alpha_1,\dots,\boldsymbol{a}-\boldsymbol{b}+\alpha_{r_2-1},\boldsymbol{a}-\boldsymbol{b})}F_2 \otimes$$
$$\otimes S_{(\boldsymbol{b}+\beta_1+\dots,\boldsymbol{b}+\beta_{r_1-1},\boldsymbol{b},-\boldsymbol{a}+\boldsymbol{b},-\boldsymbol{a}+\boldsymbol{b}-\alpha_{r_2-1},\dots,-\boldsymbol{a}+\boldsymbol{b}-\alpha_1)}F_1 \otimes S_{(0^{f_0-r_1},-\boldsymbol{b},-\boldsymbol{b}-\beta_{r_1-1},\dots,-\boldsymbol{b}-\beta_1)}F_0.$$

Here we sum over all partitions α, β and natural numbers $\boldsymbol{a}, \boldsymbol{b}$. The entries of d_2 are a representation $F_2 \otimes F_1^*$ corresponding to $\alpha = (1)$, $\beta = \boldsymbol{a} = \boldsymbol{b} = 0$, the entries of d_1 is a representation $F_1 \otimes F_0^*$ corresponding to $\beta = (1)$, $\alpha = \boldsymbol{a} = \boldsymbol{b} = 0$, the entries of a_3 are a representation $\bigwedge^{r_2} F_2 \otimes \bigwedge^{r_2} F_1^*$ corresponding to $\alpha = \beta = \boldsymbol{b} = 0$, $\boldsymbol{a} = 1$, and the entries of a_2 are a representation $\bigwedge^{r_2} F_2^* \otimes \bigwedge^{r_1+r_2} F_1 \otimes \bigwedge^{r_1} F_0^*$ corresponding to $\alpha = \beta = \boldsymbol{a} = 0$, $\boldsymbol{b} = 1$.

The defining relations in the ring R_a can be made explicit. They involve usual straightening relations between the minors of differentials d_2, d_1, Plücker relations between the Buchsbaum-Eisenbud multipliers and additional relations between minors of d_2 (resp. d_1) and Buchsbaum-Eisenbud multipliers corresponding to Garnir relations on two columns, when one of the columns corresponds to Buchsbaum-Eisenbud multipliers. There relations are described in detail in [40], [42].

15. The rings R_a generated by Buchsbaum-Eisenbud multipliers.

In this section we recall properties of the rings R_a which are obtained from coordinate rings of the varieties of generic complexes by adding the Buchsbaum-Eisenbud multipliers and factoring the relations satisfied by them. These rings are the starting point of our construction. Their properties (rational singularities and sphericality) are essential for the whole approach. Most of the results of this section were proved in [40], section 1. The additional results are easy consequences.

In what follows we use heavily representation theory of GL_n. For a GL_n dominant weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ we denote $S_\lambda F = S_{(\lambda_1,\lambda_2,\dots,\lambda_n)}F$ *the Schur functor* on the space $F = K^n$. For the purpose of this paper *a spherical variety* is a variety with a linearly reductive group action whose coordinate ring is a multiplicity free representation of this group.

Let us fix the format $(f_0, \dots, f_n)$. We work over a fixed field K. In this section we assume that K has characteristic zero. Consider the variety X_c of complexes

$$0 \to F_n \xrightarrow{d_n} F_{n-1} \to \dots \to F_1 \xrightarrow{d_1} F_0$$

of vector spaces over K, with *rank* $F_i = f_i$ and *rank* $d_i \le r_i$. We fix bases $\{e^i_{j(i)}\}_{1\le j(i)\le f_i}$ of F_i for each $i = 1, \dots, n$.

The coordinate ring R_c of X_c can be obtained as follows. We add to K the variables $X^{(i)}_{j(i),k(i)}$ $(1 \le i \le n, 1 \le j(i) \le f_{i-1}, 1 \le k(i) \le f_i)$ which are the entries of the generic maps d_i in bases $\{e^i_{j(i)}\}_{1\le j(i)\le f_i}$. The corresponding $f_{i-1} \times f_i$ matrix of variables over S_c is denoted $X^{(i)}$. We denote the resulting polynomial ring S_c. We define the ideal J_c in the polynomial ring S_c as follows. J_c is generated by the entries of matrices $X^{(i-1)} \circ X^{(i)}$ $(2 \le i \le n)$ and by the $(r_i + 1) \times (r_i + 1)$ minors of $X^{(i)}$ $(1 \le i \le n - 1)$. Finally we define $R_c := S_c/J_c$. The ring R_c is the coordinate ring of X_c which is *the variety of generic complexes.*

Recall that the variety X_c has a natural desingularization Z_c. For $i=1,\ldots,n-1$ denote by $Grass(r_{i+1},F_i)$ the Grassmannian of subspaces of rank r_{i+1} of F_i. Let

$$0\rightarrow\mathcal{R}_i\rightarrow F_i\times Grass(r_{i+1},F_i)\rightarrow\mathcal{Q}_i\rightarrow 0$$

be the tautological sequence on $Grass(r_{i+1},F_i)$.

$$Z_c=\{((d_1,\ldots,d_n),(R_1,\ldots,R_{n-1}))\in X_c\times\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)\mid Im(d_{i+1})\subset R_i\ \}.$$

Denote $p:Z_c\rightarrow\prod_{i-1}^{n-1}$, $q:Z_c\rightarrow X_c$ the natural projections. We have $p_*\mathcal{O}_{Z_c}=\otimes_{i=1}^n Sym(\mathcal{Q}_i\otimes\mathcal{R}_{i-1}^*)$, where $\mathcal{Q}_n=F_n$.

Theorem 15.1. *(*[16], [40]*) The variety X_c carries the natural action of the group $GL:=\prod_{i=0}^n GL(F_i)$. It is a spherical variety and it has rational singularities. The coordinate ring R_c has a multiplicity free decomposition to the irreducible representations of GL given by the formula*

$$R_c=\oplus_{\alpha^{(1)},\ldots,\alpha^{(n)}}\otimes_{i=0}^n S_{(\alpha_1^{(i)},\ldots,\alpha_{r_i}^{(i)},-\alpha_{r_{i+1}}^{(i+1)},\ldots,-\alpha_1^{(i+1)})}F_i$$

where we sum over all n-tuples $(\alpha^{(1)},\ldots,\alpha^{(n)})$ of partitions, with the i-th partition $\alpha^{(i)}=(\alpha_1^{(i)},\ldots\alpha_{r_i}^{(i)})$ having at most r_i parts. Here by convention $\alpha^{(n+1)}=0$ has no parts and $\alpha^{(0)}=(0^{f_0-r_1})$ has f_0-r_1 parts.

This result follows by standard methods from Cauchy decomposition

$$\otimes_{i=1}^n Sym(\mathcal{Q}_i\otimes\mathcal{R}_{i-1}^*)=\oplus_{\alpha^{(1)},\ldots,\alpha^{(n)}}\otimes_{i=1}^n S_{\alpha^{(i)}}\mathcal{Q}_i\otimes S_{\alpha^{(i)}}\mathcal{R}_{i-1}^*,$$

and the fact that by Bott theorem the higher cohomology of the above sheaf vanishes and the sections decompose as given in the Theorem 15.1.

We have *a generic complex $\mathbb{F}_\bullet^c$ of format* $(f_0,\ldots,f_n)$ defined over the ring R_c. It is a complex

$$\mathbb{F}_\bullet^c:0\rightarrow F_n\otimes R_c\xrightarrow{d_n}F_{n-1}\otimes R_c\rightarrow\ldots\rightarrow F_1\otimes R_c\xrightarrow{d_1}F_0\otimes R_c$$

with d_i given (in our bases of F_i) by the matrix $X^{(i)}$.

In [40], section 1 we carried a similar procedure for the rings R_a.

Consider the affine space $X=\prod_{i=1}^n Hom_K(F_i,F_{i-1})\times\prod_{i=1}^{n-1}\bigwedge^{r_i}F_{i-1}$. The coordinates in $Hom_K(F_i,F_{i-1})$ are the entries of the map d_i and the coordinates of $\bigwedge^{f_i}F_{i-1}$ are the Buchsbaum-Eisenbud multipliers a_i. We also consider the analogue of the desingularization Z_c.

$$Z_a\subset X\times\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)$$

Definition 15.2. *The variety Z_a,*

$$Z_a\subset X\times\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)$$

is defined by the following conditions. A point

$$\{(d_1,\ldots,d_n;a_1,\ldots,a_{n-1}),(R_1,\ldots,R_{n-1})\}\in X\times\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)$$

is in Z_a if and only if the following conditions are satisfied.

(1) *$Im(d_i)\subset R_i\subset Ker(d_{i-1})$,*
(2) *$a_i\in\bigwedge^i R_{i-1}$,*
(3) *For the induced map $d'_i: Q_i\to R_{i-1}$ we have $d'_n=a_n$, $d'_i=a_{i+1}a_i$ for $i=1,\ldots,n-1$. We denote $p: Z_a\to X$, $q: Z_a\to\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)$ two projections, and we define the variety $X_a:=p(Z_a)\subset X$.*

The variety Z_a is fiber bundle over $\prod_{i=1}^{n-1}Grass(r_{i+1},F_i)$, and the fibre over a point $(R_1,\ldots,R_{n-1})$ is the affine variety given by the general $r_i\times r_i$ matrices $d'_i: Q_i\to R_{i-1}$ and elements $a_i\in\bigwedge^{r_i}R_{i-1}$ satisfying relations (3).

It turns out that Z_a has rational singularities so it can be used in a similar way to Z_c. The following result is proved in [40], section 1.

Theorem 15.3. *The variety X_a carries the natural action of the group $GL:=\prod_{i=0}^{n}GL(F_i)$. It is a spherical variety and it has rational singularities. The coordinate ring R_a of X_a has a multiplicity free decomposition to the irreducible representations of GL given by the formula*

$$R_a=\oplus_{\alpha^{(1)},\ldots,\alpha^{(n)},x^{(1)},\ldots,x^{(n)}}$$
$$\otimes_{i=0}^{n}S_{(\chi^{(i)}+\alpha_1^{(i)},\ldots,\chi^{(i)}+\alpha_{r_i-1}^{(i)},\chi^{(i)},-\chi^{(i+1)},-\chi^{(i+1)}-\alpha_{r_{i+1}-1}^{(i+1)},\ldots,-\chi^{(i+1)}-\alpha_1^{(i+1)})}F_i.$$

Here the notation is as follows. We sum over all n-tuples $(\alpha^{(1)},\ldots,\alpha^{(n)})$ of partitions, with the i-th partition $\alpha^{(i)}=(\alpha_1^{(i)},\ldots\alpha_{r_i-1}^{(i)})$ having at most r_i-1 parts. Here by convention $\alpha^{(n+1)}=0$ has no parts and $\alpha^{(0)}=(0^{f_0-r_1})$ has f_0-r_1 parts.

We also sum over n-tuples of natural numbers $x^{(1)},\ldots,x^{(n)}$ (degrees with respect to the a_i's). The numbers $\chi^{(i)}$ are partial Euler characteristics and they are

$$\chi^{(i)}=\sum_{j=1}^{i}(-1)^{i-j}x^{(j)}.$$

Note that $\chi^{(i)}+\chi^{(i+1)}=x^{(i+1)}$, so all the weights listed in the formula are dominant.

The defining relations of the ring R_a are written down explicitly in [40], section 1. The structure of R_a can be described also in a characteristic free way (replacing K by $\mathbb{Z}$ and using filtrations instead of direct sums). This was done in [40], section 1 and in [42] where some errors in characteristic free part of the approach were fixed.

We denote by $\mathbb{F}^a_\bullet$ the complex $\mathbb{F}^c_\bullet\otimes_{R_c}R_a$. This complex has a weaker universality property, true even in a characteristic free version.

Theorem 15.4. *The complex $\mathbb{F}^a_\bullet$ is the universal complex of format $(f_0,\ldots,f_n)$ which is acyclic in codimension 1. This means that for every pair $(S,\mathbb{G})$ such that S is a Noetherian ring and $\mathbb{G}$ is a complex of free modules of format $(f_0,\ldots,f_n)$ over S which is acyclic of codimension 1 (i.e. the set of points where the complex is not acyclic has a defining ideal of depth ≥ 2), then there is a unique homomorphism $\phi: R_a\to S$ such that $\mathbb{G}=\mathbb{F}^a_\bullet\otimes_{R_a}S$.*

This has the following consequence (which goes back to Hochster ([23]) and is even true over $\mathbb{Z}$).

Corollary 15.5. *For $n = 2$ the pair $(R_a, \mathbb{F}^a_\bullet)$ is a generic acyclic complex for the format (f_0, f_1, f_2).*

In the remainder of this section we look more closely at the homology modules of the complex $\mathbb{F}^a_\bullet$. These modules are possible to analyze thanks to the multiplicity free structure of the ring R_a.

We look at the module $F_j \otimes R_a$ and compare it to the modules $F_{j+1} \otimes R_a$ and $F_{j-1} \otimes R_a$. We describe the cancellations that occur between them when applying the map d_{j+1} and d_j. Looking at the representations $F_j \otimes R_a$ and $F_{j-1} \otimes R_a$, let us assume that they have common representations coming from summands

$$F_j \otimes [\otimes_{i=0}^n S_{(\chi^{(i)}+\alpha_1^{(i)},\dots,\chi^{(i)}+\alpha_{r_i-1}^{(i)},\chi^{(i)},-\chi^{(i+1)},-\chi^{(i+1)}-\alpha_{r_{i+1}-1}^{(i+1)},\dots,-\chi^{(i+1)}-\alpha_1^{(i+1)})}F_i]$$

and

$$F_{j-1} \otimes [\otimes_{i=0}^n S_{(\psi^{(i)}+\beta_1^{(i)},\dots,\psi^{(i)}+\beta_{r_i-1}^{(i)},\psi^{(i)},-\psi^{(i+1)},-\psi^{(i+1)}-\beta_{r_{i+1}-1}^{(i+1)},\dots,-\psi^{(i+1)}-\beta_1^{(i+1)})}F_i]$$

We denote the degree of the first (resp. the second) representation with respect to the structure map a_i by $x^{(i)}$ (resp. $y^{(i)}$).

We note that by Pieri formula we have two possibilities. On the coordinate F_j we can add a box to first $r_j - 1$ entries, or to the r_j-th entry.

It is easy to see that when we add a box in $F_j \otimes R_a$ to one of the first $r_j - 1$ places, we can always find a corresponding representation in $F_{j-1} \otimes R_a$, with $\alpha^{(i)} = \beta^{(i)}$, $\chi^{(i)} = \psi^{(i)}$ for $i = 1, \dots, n$.

Consider the crucial case when we add a box in $F_j \otimes R_a$ to the r_j-th place, and we add a box in $F_{j-1} \otimes R_a$ in the $(r_{j-1} + 1)$'st place. Then comparing the weights we have:

$$\chi^{(i)} = \psi^{(i)}, \alpha^{(i)} = \beta^{(i)}, \forall\ i \neq j,$$

$$\chi^{(j)} + 1 = \psi^{(j)}, \alpha_k^{(j)} = \beta_k^{(j)} - 1, \forall\ 1 \le k \le r_j - 1.$$

This translates to $x^{(i)} = y^{(i)}\ \forall i \neq j, j+1$, and $x^{(j)} = y^{(j)} + 1, x^{(j+1)} = y^{(j+1)} - 1$.

This means such cancellation cannot occur when $y^{(j+1)} = 0$, so the corresponding representations stay in $H_{j-1}(\mathbb{F}^a_\bullet)$. Notice that this does not happen for $j - 1 = n, n-1$, as there is no $y^{(j+1)}$ in such cases.

Theorem 15.6. *The homology groups $H_n(\mathbb{F}^a_\bullet) = H_{n-1}(\mathbb{F}^a_\bullet) = 0$. For $1 \le j - 1 \le n - 2$ we have*

$$H_{j-1}(\mathbb{F}^a_\bullet) = \oplus_{\beta^{(1)},\dots,\beta^{(n)},y^{(1)},\dots,y^{(n)},y^{(j+1)}=0}$$

$$\otimes_{i=0}^{j-2} S_{(\psi^{(i)}+\beta_1^{(i)},\dots,\psi^{(i)}+\beta_{r_i-1}^{(i)},\psi^{(i)},\psi^{(i+1)},-\psi^{(i+1)}-\beta_{r_{i+1}-1}^{(i+1)},\dots,-\psi^{(i+1)}-\beta_1^{(i+1)})}F_i \otimes$$

$$\otimes S_{(\psi^{(j-1)}+\beta_1^{(j-1)},\dots,\psi^{(j-1)}+\beta_{r_{j-1}-1}^{(j-1)},\psi^{(j-1)},1-\psi^{(j)},-\psi^{(j)}-\beta_{r_j-1}^{(j)},\dots,-\psi^{(j)}-\beta_1^{(j)})}F_{j-1} \otimes$$

$$\otimes_{i=j}^n S_{(\psi^{(i)}+\beta_1^{(i)},\dots,\psi^{(i)}+\beta_{r_i-1}^{(i)},\psi^{(i)},-\psi^{(i+1)},-\psi^{(i+1)}-\beta_{r_{i+1}-1}^{(i+1)},\dots,-\psi^{(i+1)}-\beta_1^{(i+1)})}F_i.$$

In particular the homology group $H_{j-1}(\mathbb{F}^a_\bullet)$ is annihilated by the ideal $I(a_{j+1})$ generated by the entries of the $(j+1)$'st Buchsbaum-Eisenbud multiplier map.

Proof. To prove the theorem it is enough to show that the indicated cancellations indeed occur. This is not difficult since for each i the module $F_i \otimes R_a$ is multiplicity free as a $\mathrm{GL}(\mathbb{F})$-module. Moreover the highest weight vectors of irreducible representations are not difficult to write down as only the Pieri formula of multiplying by F_i is involved. We skip the details here since the result is not used elsewhere. $\square$

Let us look at the generator of $H_{j-1}(\mathbb{F}^a_\bullet)$ for $n-2 \geq j-1 \geq 1$. It will be a minimal partition in $H_{j-1}(\mathbb{F}^a_\bullet)$. We obtain it by setting $\beta^{(i)} = 0$ for $i = 1, \ldots, n$, $\psi^{(i)} = 0$ for $i > j-1$, $\psi^{(i)} = (-1)^{j-1-i}$ for $1 \leq i \leq j-1$. The resulting representation is (up to maximal exterior powers of F_i) $\bigwedge^{r_{j-1}+1} F_{j-1}$. The existence of such cycle $q_1^{(j-1)}$ means in the generic ring one will need to add a representation $F_j^* \otimes \bigwedge^{r_{j-1}+1} F_{j-1}$, corresponding to the map $p_1^{(j-1)} : \bigwedge^{r_{j-1}+1} F_{j-1} \otimes R \to F_j \otimes R$ covering this cycle. The maps $p_1^{(j-1)}$ are the same as the maps b_{j-1} coming from the Second Structure Theorem of Buchsbaum and Eisenbud ([10], section 6).

Finally we note the key property of the lattice of weights of R_a.

Remark 15.7. *Let Λ be the lattice of highest weights of the ring R_a. Let Λ_{even} (resp. Λ_{odd}) be the projection of the weight of $\mathrm{GL}(\mathbb{F})$ onto the weight of $\mathrm{GL}(\mathbb{F}_{even})$ (resp. $\mathrm{GL}(\mathbb{F}_{odd})$), where*

$$\mathrm{GL}(\mathbb{F})_{even} = \prod_{i\ even} \mathrm{GL}(F_i), \mathrm{GL}(\mathbb{F})_{odd} = \prod_{i\ odd} \mathrm{GL}(F_i).$$

Then the projections $\Lambda \to \Lambda_{even}$ (resp. $\Lambda \to \Lambda_{odd}$) are isomorphisms. In other words every weight in R_a is uniquely determined by its even and odd parts.

Let us specialize to the case $n = 3$. We will use slightly different notation.

The incidence variety Y_a giving a modification of variety $X_a := Spec\ R_a$.

Y_a is a subset of $X_a \times Grass(r_3, F_2) \times Grass(r_2, F_1) \times Gras(r_1, F_0)$ consisting of tuples $((d_3, d_2, d_1, a_2, a_1), R_2, R_1, R_0)$ such that

(1) $(d_3, d_2, d_1, a_2, a_1) \in X_a$,
(2) $Im(a_i) \subset \bigwedge^{r_i} R_i, Im\ d_i \subset R_i \subset Ker\ d_{i-1}$ for $i = 0, 1, 2$,
(3) For the induced maps $d'_i : Q_i := F_i/R_i \to R_{i-1}$ and $a'_i \in \bigwedge^{r_i} R_i$, we have $det(d'_3) = a'_3$, $det(d'_2) = a'_3 a'_2$, $det(d'_1) = a'_2 a'_1$.

One has natural projections $p_a : Y_a \to X_a$, $q_a : Y_a \to Grass$ where $Grass := Grass(r_3, F_2) \times Grass(r_2, F_1) \times Gras(r_1, F_0)$.

One gets the decomposition of R_a ([40], section 1, formula (10) in [47])

Proposition 15.8. *We have*

$$R_a = \oplus_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\alpha,\beta,\gamma} S_{(\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_1,\ldots,\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_{r_3-1},\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c})} F_3 \otimes$$

$$\otimes S_{(\boldsymbol{b}-\boldsymbol{c}+\beta_1,\ldots,\boldsymbol{b}-\boldsymbol{c}+\beta_{r_2-1},\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_{r_3-1},\ldots,-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_1)} F_2 \otimes$$

$$\otimes S_{(\boldsymbol{c}+\gamma_1,\ldots,\boldsymbol{c}+\gamma_{r_1-1},\boldsymbol{c},\boldsymbol{c}-\boldsymbol{b},\boldsymbol{c}-\boldsymbol{b}-\beta_{r_2-1},\ldots,\boldsymbol{c}-\boldsymbol{b}-\beta_1)} F_1 \otimes$$

$$\otimes S_{(0^{r_0},-\boldsymbol{c},-\boldsymbol{c}-\gamma_{r_1-1},\ldots,-\boldsymbol{c}-\gamma_1)} F_0.$$

where we sum over all triples of natural numbers $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and triples of partitions α, β, γ such that $\alpha'_1 < r_3$, $\beta'_1 < r_2$ and $\gamma'_1 < r_1$.

Corollary 15.9. *The ring R_a is a multiplicity free representation for the action of the group $\prod_{i=0}^{3} GL(F_i)$, so the variety X_a is spherical.*

In the case of $n=3$ we have over the ring R_a *depth* $I(d_1)=1$, *depth* $I(d_2)=$ *depth* $I(d_3)=2$. In order to get acyclicity of $\mathbb{F}^a_\bullet$ it is enough to raise the depth of $I(d_3)$ to 3. This can be done by killing the cycles in $\mathbb{F}^a_\bullet$, in the Koszul complex on $I(d_3)$ and killing the higher direct image $R^1 j_*(\mathcal{O}_{U_a})$ where $U_a = X_a \setminus V(I(d_3))$ and $j: U_a \to X_a$ is an inclusion. By the results of section 12 we know that vanishing of first homology groups of both complexes and of $R^1 j_*(\mathcal{O}_{U_a})$ is equivalent.

We denote

$$\mathcal{K}_a:\ 0 \to \bigwedge^0 \mathcal{K} \to \bigwedge^1 \mathcal{K} \to \bigwedge^2 \mathcal{K} \to \bigwedge^3 \mathcal{K}$$

the beginning of the Koszul complex on $I(d_3)$, the ideal of maximal minors of d_3. Thus $\mathcal{K} := \bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2 \otimes_{\mathbb{C}} R_a$. We treat $\mathcal{K}_a$ as a complex concentrated in degrees, 0 to 3 with differential of degree -1.

Proposition 15.10. *We have the following isomorphisms.*

1. $H_1(\mathbb{F}^a_\bullet) = Ker(\bigwedge^3 \mathcal{K} \otimes R^1 j_*(\mathcal{O}_{U_a}) \stackrel{d_3\otimes 1}{\to} \bigwedge^2 \mathcal{K} \otimes R^1 j_*(\mathcal{O}_{U_a}))$.
2. $H_1(\mathcal{K}_a) = Ker(F_3 \otimes R^1 j_*(\mathcal{O}_{U_a}) \stackrel{d_3\otimes 1}{\to} F_2 \otimes R^1 j_*(\mathcal{O}_{U_a}))$, *i.e. it is the set of elements in* $R^1 j_*(\mathcal{O}_{U_a})$ *annihilated by* $I(d_3)$.

Let us identify the generator of $H_1(\mathcal{K}_a)$.

Proposition 15.11. *The module $H_1(\mathcal{K}_a)$ is generated by the image of the map*

$$q_1 : F_3^* \otimes \bigwedge^{r_1+1} F_1 \to \bigwedge^2 (\bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2) \otimes R_a =$$

$$= S_{2^{r_3}} F_3^* \otimes \bigwedge^2 (\bigwedge^{r_3} F_2) \otimes R_a.$$

There is the only one (up to nonzero scalar) nonzero equivariant map of this type.

In terms of the formula it is given by

$$h_t^* \otimes f_{i_1} \wedge \ldots \wedge f_{i_{r_1+1}} \mapsto \sum_{J,K} u_{I,J,K}\, g_J \otimes g_K$$

where the coefficient $u_{I,J,K}$ is given by the formula

$$\sum \pm \langle 1,\ldots,\hat{t},\ldots,r_3 | j_1 \ldots, \hat{j}_s, \ldots, j_{r_3}\rangle_3 \langle (j_s, k_1, \ldots, k_{r_3})' | (i_1, \ldots, i_{r_1+1})'\rangle_2.$$

Here $I=(i_1,\ldots,i_{r_1+1})$, $J=(j_1,\ldots,j_{r_3})$, $K=(k_1,\ldots,k_{r_3})$, and J' denotes the complement of the set J. Also $\langle|\rangle_i$ denotes the minors of d_i for $i=2,3$.

Proof. Let us look at possible equivariant maps q_1 as stated in the Proposition 15.11. Looking at the weight corresponding to $\mathrm{GL}(F_1)$ we see that the only way such a map can occur is for the summand in R_a having $\boldsymbol{b}=\boldsymbol{c}=0, \gamma=0$ and $\beta_1=\ldots=\beta_{r_2-1}=1$. Looking at the weight of $\mathrm{GL}(F_2)$ we need a trivial $SL(F_2)$ representation in $\bigwedge^2(\bigwedge^{r_3} F_2) \otimes S_{(1^{r_2-1},0^2,(-1)^{r_3-1})} F_2$. It can happen only once, choosing the representation $S_{2^{r_3-1},1^2)} F_2$ in $\bigwedge^2(\bigwedge^{r_3} F_2)$. Similar reasoning shows that the representation $F_3^* \otimes \bigwedge^{r_1+1} F_1$ cannot occur in $\bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2 \otimes R_a$.

Indeed, looking at the weight of $\mathrm{GL}(F_1)$ we see again that we need $\boldsymbol{b} = \boldsymbol{c} = \gamma = 0$ and $\beta_1 = \ldots = \beta_{r_2-1} = 1$. Then looking at the weight of $\mathrm{GL}(F_2)$ we get a contradiction. Finally looking at the occurrence of $F_3^* \otimes \bigwedge^{r_1+1} F_1$ in $\bigwedge^3(\bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2) \otimes R_a$ we see it cannot happen, as the representation $S_{(3^{r_3-1},2,1)}F_2$ does not occur in $\bigwedge^3(\bigwedge^{r_3} F_2)$.

There is another way to see that there is a cycle q_1 of the required form and that it generates $H_1(\mathcal{K}_a)$.

We can calculate the higher direct image of $\mathcal{O}_{Z_a}$ with the map a_3' inverted. This is done using Bott Theorem, applied to the quadruples of weights (corresponding respectively to F_3, F_2, F_1, F_0):

$$\begin{gathered}((\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_1,\ldots,\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_{r_3-1},\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}),\\ (\boldsymbol{b}-\boldsymbol{c}+\beta_1,\ldots,\boldsymbol{b}-\boldsymbol{c}+\beta_{r_2-1},\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_{r_3-1},\ldots,-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_1),\\ (\boldsymbol{c}+\gamma_1,\ldots,\boldsymbol{c}+\gamma_{r_1-1},\boldsymbol{c},\boldsymbol{c}-\boldsymbol{b},\boldsymbol{c}-\boldsymbol{b}-\beta_{r_2-1},\ldots,\boldsymbol{c}-\boldsymbol{b}-\beta_1),\\ (0^{r_0},-\boldsymbol{c},-\boldsymbol{c}-\gamma_{r_1-1},\ldots,-\boldsymbol{c}-\gamma_1))\end{gathered}$$

where we sum over all partitions $\alpha, \beta, \gamma, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{N}$ and $\boldsymbol{a} \in \mathbb{Z}$, and then calculate the homology by Bott Theorem. We see that in $R^1 j_*(\mathcal{O}_{U_a})$ we get the required representation for

$$\boldsymbol{a} = -2, \boldsymbol{b} = \boldsymbol{c} = \gamma = 0, \beta_1 = \ldots = \beta_{r_2-1} = 1, \alpha_1 = \ldots = \alpha_{r_3-1} = 1.$$

Moreover, if we increase $\boldsymbol{a}$ by one to -1, there will be no corresponding representation in $R^1 j_*(\mathcal{O}_{U_a})$, so our representation is annihilated by $I(d_3)$, so it gives an element in $H_1(\mathcal{K}_a)$.

The formula giving q_1 can be deduced from analyzing the way the equivariant map q_1 was constructed. Looking at the summand of R_a we used it is clear it has to involve the products of $(r_3-1) \times (r_3-1)$ minors of d_3 and of $(r_2-1) \times (r_2-1)$ minors of d_2. $\square$

Remark 15.12. *Notice that we used the* $\mathrm{SL}(F_3) \times \mathrm{SL}(F_1)$ *equivariance instead of* $\mathrm{GL}(F_3) \times \mathrm{GL}(F_1)$ *equivariance to construct the map* q_1*. It is caused by the fact that under the exact identification of weights in the Lie algebra of type* $T_{p,q,r}$ *with the weights of* $\mathrm{GL}(F_3) \times \mathrm{GL}(F_1)$ *there is a copy of line bundle which centralizes* $\mathfrak{g}_0$ *which acts in a nontrivial way.*

Let us compare the elements $q^{(1)}$ and q_1 as elements of $F_3 \otimes R^1 j_*(\mathcal{O}_{U_a})$ and $R^1 j_*(\mathcal{O}_{U_a})$ respectively. Representation theory and Bott theorem show that they are related as follows.

The map $q^{(1)}$ can be expressed as a composition

$$\bigwedge^{r_1+1} F_1 \otimes M_3^{-1} \otimes M_2 \otimes M_1^{-1} \xrightarrow{tr\otimes 1} F_3 \otimes F_3^* \otimes \bigwedge^{r_1+1} F_1 \otimes M_3^{-1} \otimes M_2 \otimes M_1^{-1} \to$$

$$\xrightarrow{q_1 \otimes 1} F_3 \otimes R^1 j_*(\mathcal{O}_{U_a}).$$

Conversely, the map q_1 can be expressed in terms of $q^{(1)}$ as follows

$$F_3^* \otimes \bigwedge^{r_1+1} F_1 \otimes M_3^{-1} \otimes M_2 \otimes M_1^{-1} \xrightarrow{1\otimes q^{(1)}} F_3^* \otimes F_3 \otimes R^1 j_*(\mathcal{O}_{U_a}) \xrightarrow{ev\otimes 1} R^1 j_*(\mathcal{O}_{U_a}).$$

This follows from the fact that $R^1 j_*(\mathcal{O}_{U_a})$ is multiplicity free.

We conclude that adding to R_a the entries of the cycle b killing $q^{(1)}$ and cycle p_1 killing q_1, and performing the ideal transform with respect to $I(d_3)$ results in the same ring R_1.

We denote $X_1 = Spec(R_1)$ and $U_1 = X_1 \setminus V(I(d_3))$. Over the open set U_a these rings are isomorphic to $\mathcal{O}_{U_a}$ with the variables corresponding to the first defect $F_3^* \otimes \bigwedge^{r_1+1} F_1$.

16. The structure maps p_i.

Let $\mathbb{F}_\bullet$ be an acyclic complex of length three over a ring R. Let $\mathbb{L} := \mathbb{L}(r_1+1, F_3, F_1)$ be the corresponding defect algebra. Finally, let

$$0 \to \bigwedge^0 \mathcal{K} \to \bigwedge^1 \mathcal{K} \to \bigwedge^2 \mathcal{K} \to \bigwedge^3 \mathcal{K}$$

be the beginning of the Koszul complex on $I(d_3)$, the ideal of maximal minors of d_3. Thus $\mathcal{K} := \bigwedge^{r_3} F_3^* \otimes \bigwedge^{r_3} F_2 \otimes_{\mathbb{C}} R$.

In section 15 we constructed the map $p_1 : \mathbb{L}_1^* \to \bigwedge^1 \mathcal{K}$ covering the cycle q_1. We continue to construct the higher maps p_i $(i \geq 2)$.

Proposition 16.1. *Let $\mathbb{F}_\bullet$ be an acyclic complex of format (f_0, f_1, f_2, f_3). There exists a structure map p_2 making the following diagram commute.*

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^0 \mathcal{K} & \to & \bigwedge^1 \mathcal{K} & \to & \bigwedge^2 \mathcal{K} & \to & \bigwedge^3 \mathcal{K} \\ & & & & \uparrow p_2 & & \uparrow \bigwedge^2(p_1) & & \\ & & 0 & \to & \mathbb{L}_2^* & \to & (\bigwedge^2 \mathbb{L}_1)^* & & \end{array}$$

Proof. Since the upper row is an exact sequence, it is enough to check that the composition of the Koszul differential with $\bigwedge^2(p_1)$ restricted to the image of $\mathbb{L}_2^*$ is zero. This calculation is carried out in Theorem 2.9 from [47] (the map p_1 is denoted there by $b^{\#}$). Alternatively, one can just prove it for a split complex (with arbitrary choice of p_1 involving defect variables). This is left to the reader. $\square$

The defect of the map p_2 is equal to $\mathbb{L}_2$.

The definition of the defect algebra allows to introduce the higher maps p_i.

Theorem 16.2. [47] *Let $\mathbb{F}_\bullet$ be an acyclic complex of format (f_0, f_1, f_2, f_3).*

There exists a sequence of structure maps $p_i : \mathbb{L}_i^* \to \bigwedge^1 \mathcal{K}$ satisfying the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^0 \mathcal{K} & \to & \bigwedge^1 \mathcal{K} & \to & \bigwedge^2 \mathcal{K} & \to & \bigwedge^3 \mathcal{K} \\ & & & & \uparrow p_{m+1} & & \uparrow q_{2,m+1} & & \uparrow q_{3,m+1} \\ & & 0 & \to & \mathbb{L}_{m+1}^* & \to & (\bigwedge^2 \mathbb{L})_{m+1}^* & \to & (\bigwedge^3 \mathbb{L})_{m+1}^* \end{array}$$

where $q_{2,m+1} = \sum (p_i \wedge p_j)$, $q_{3,m+1} = \sum (p_i \wedge p_j \wedge p_k)$.

Proof. The upper row is an exact complex, and the diagram commutes by the definition of the Koszul differential and the maps in the lower row. The result follows by an elementary diagram chase. $\square$

The relation with the defect Lie algebra is that the defect (i.e. non-uniqueness) of each map p_m is equal to $\mathbb{L}_m$. Defect refers to the fact that p_{m+1} is a lifting of certain cycles and we can modify p_m by the map from $\mathbb{L}_m^*$ to R, i.e. by an element of $\mathbb{L}_m$.

The idea of the construction of the generic ring carried out in [47] is to build it up taking these symmetries into account. More precisely, define R_m to be the ring we obtain from R_a by adding generically the coefficients of the maps $p_1, \ldots, p_m$, and then dividing by the appropriate relations (those that vanish when specializing to a splitting complex with arbitrary choice of the maps $p_1, \ldots, p_m$, compare Lemma 2.4 [47]), and take the ideal transform with respect to $I(d_2)I(d_3)$. We get the action of the Lie algebra $\mathbb{L}/(\sum_{j>m} \mathbb{L}_j)$ on such ring R_m (Theorem 2.12, [47]).

Definition 16.3. *We define $\hat{R}_{gen} := lim_m R_m$, $\mathbb{F}^{gen}_\bullet = \mathbb{F}^a_\bullet \otimes_{R_a} \hat{R}_{gen}$.*

Similarly, for every m we have a diagram

$$\begin{array}{ccccc} U_m := Y_m \setminus p_m^{-1}(D_3) & \stackrel{j'_m}{\to} & Y_m & \stackrel{q_{gen}}{\to} & Grass \\ \downarrow\ p'_m & & \downarrow\ p_m & & \\ X_m \setminus D_3 & \stackrel{j_m}{\to} & X_m & & \end{array}$$

so finally, after including all p_i's we get a diagram

$$\begin{array}{ccccc} U_{gen} := Y_{gen} \setminus p_{gen}^{-1}(D_3) & \stackrel{j'_{gen}}{\to} & Y_{gen} & \stackrel{q_{gen}}{\to} & Grass \\ \downarrow\ p'_{gen} & & \downarrow\ p_{gen} & & \\ X_{gen} \setminus D_3 & \stackrel{j_{gen}}{\to} & X_{gen} & & \end{array}$$

Our goal is to show that $\mathcal{R}^1(j_{gen})_*\mathcal{O}_{X_{gen}\setminus D_3} = 0$ proving that the complex $\mathbb{F}^{gen}_\bullet := \mathbb{F}^a_\bullet \otimes_{R_a} \mathcal{O}_{X_{gen}}$ is the generic complex.

Remark 16.4. *Two observations will be useful in the future.*

(1) The set $U_{gen} := Y_{gen} \setminus p_{gen}^{-1}(D_3)$ has a simple geometric interpretation. It is isomorphic to $U_0 := X_a \setminus p_a^{-1}(D_3) \times \oplus_{i>0}\mathbb{L}_i$. Indeed, if the map d_3 splits, then each map p_i splits into its defect and a map defined uniquely. Moreover, the affine space $\oplus_{i>0}\mathbb{L}_i$ is clearly isomorphic to the open Schubert cell in the homogeneous space $\mathcal{G}/\mathcal{P}$ where $\mathcal{G}$ is the Kac-Moody group associated to the graph $T_{p,q,r}$ and $\mathcal{P}$ is the parabolic associated to the simple root corresponding to the vertex z_1.

(2) The rings R_m (and therefore the ring R_{gen}) are domains. Indeed, by construction the depth of the ideal $I(d_3)$ in these rings is ≥ 1 and after inverting an $r_3 \times r_3$ minor D of d_3 we get a polynomial ring over $R_a[D^{-1}]$.

17. The spectral sequence and the complexes $\mathcal{K}(\alpha, \beta, s)_\bullet$ over $U(\mathbb{L})$.

The next step (section 3 in [47]) is the analysis of the spectral sequence which allows us to calculate the cohomology of $\mathcal{F}_{gen} := q_{gen*}(\mathcal{O}_{Y_{gen}\setminus D_3})$ (see notation preceding Remark 16.4). The spectral sequence is equivariant with respect to the group $\prod_{i=0}^3 GL(F_i)$. However the representations occurring in $U(\mathbb{L})$ do not contain the representations of F_0 and they contain only the maximal exterior power of F_2. Thus the $U(\mathbb{L})$-module structure will be preserved on the isotypic components of the group $\mathbb{G}_{even} := SL(F_2) \otimes GL(F_0)$. Let us also denote $\mathbb{G}_{odd} := SL(F_3) \otimes SL(F_1)$.

Analyzing the isotypic components of the representations one reaches the following conclusion. The isotypic component of the cohomology of $\mathcal{F}_{gen}$ is calculated as a cohomology of a complex $\mathcal{K}(\sigma, \tau, t)$ of the form

$$0 \to K_0 \to K_1 \to K'_2 \oplus K''_2$$

where each term consists of a single irreducible representation of the group $\hat{\mathbb{GL}}_{odd} := GL(F_3)\times GL(F_1)$ tensored with $U(\mathbb{L})^*$. Dualizing we obtain the following statement.

Theorem 17.1. *([47], page 26, formula (38)) All duals of isotypic components of the spectral sequence are the complexes of the form*

$$\begin{gathered} S_{(\sigma_1+t+u,\sigma_2,\ldots,\sigma_{r_3})}F_3\otimes \\ \otimes S_{(\tau_1+t+u,\ldots,\tau_{r_1}+t+u,\tau_{r_1+1}+t,\tau_{r_1+2}+u,\tau_{r_1+3},\ldots,\tau_{r_1+r_2})}F_1^*\otimes U(\mathbb{L}) \\ \oplus S_{(\sigma_1+t,\sigma_2+s,\sigma_3,\ldots,\sigma_{r_3})}F_3\otimes S_{(\tau_1+t+s,\ldots,\tau_{r_1+1}+t+s,\tau_{r_1+2},\ldots,\tau_{r_1+r_2})}F_1^*\otimes U(\mathbb{L}) \\ \downarrow \\ S_{(\sigma_1+t,\sigma_2,\ldots,\sigma_{r_3})}F_3\otimes S_{(\tau_1+t,\ldots,\tau_{r_1+1}+t,\tau_{r_1+2},\ldots,\tau_{r_1+r_2})}F_1^*\otimes U(\mathbb{L}) \\ \downarrow \\ S_{(\sigma_1,\sigma_2,\ldots,\sigma_{r_3})}F_3\otimes S_{(\tau_1,\ldots,\tau_{r_1+1},\tau_{r_1+2},\ldots,\tau_{r_1+r_2})}F_1^*\otimes U(\mathbb{L}). \end{gathered}$$

Here the numbers u and s are uniquely determined by the triple (σ,τ,t) by equalities

$$\sigma_2+s=\sigma_1+1, \tau_{r_1+2}+u=\tau_{r_1+1}+1.$$

We denote the complex listed in the Theorem by $\mathcal{K}^*(\sigma,\tau,t)$.

We have the following crucial consequence.

Corollary 17.2. *([47], Theorem 3.1) Assume that all the complexes $\mathcal{K}^*(\sigma,\tau,t)$ are exact at their middle term. Then $\mathcal{R}^1(j_{gen})_*\mathcal{F}_{gen}=0$ and therefore*

$$(j_{gen})_*\mathcal{O}_{X_{gen}\setminus D_3}=H^0(Grass,\mathcal{F}_{gen})$$

and the complex $\mathbb{F}_\bullet^{gen}$ is acyclic over $(j_{gen})_*\mathcal{O}_{X_{gen}\setminus D_3}$, so it is the generic ring for our format.

In the next section we will see that the complexes are indeed exact at the middle term by identifying them with the beginning part of certain parabolic BGG resolution.

18. Main result.

In this section we draw the consequences from previous considerations. The main result is

Theorem 18.1. *For every format (f_0,f_1,f_2,f_3) there exists a generic pair*

$$(\hat{R}_{gen},\mathbb{F}_\bullet^{gen}):=((j_{gen})_*\mathcal{O}_{X_{gen}\setminus D_3},\mathbb{F}_\bullet^a\otimes_{R_a}(j_{gen})_*\mathcal{O}_{X_{gen}\setminus D_3}).$$

The generic ring $\hat{R}_{gen}$ is a general fibre of a flat family where a special fibre $\hat{R}_{spec}$ has a multiplicity free action of $\mathfrak{g}(T_{p,q,r})\times\mathfrak{gl}(F_2)\times\mathfrak{gl}(F_0)$, where $f_3=r-1, f_2=q+r, f_1=p+q, r_1=p-1$. If the algebra $\mathbb{L}(r_1+1,F_3,F_1)$ is finite dimensional, then the generic ring $\hat{R}_{gen}$ is Noetherian.

Proof. The complexes $\mathcal{K}^*(\sigma,\tau,t)$ from the Corollary 17.2. are identical to the part of BGG complex identified in Proposition 3.4. The partition α is just λ restricted to the third arm of the graph. The partition β is λ restricted to the graph A_{p+q-1} we get when we omit the third arm of the graph $T_{p,q,r}$. The number $t:=\lambda_{p+q}+1$. The differentials are the same because each component is nonzero and there is (up to a nonzero scalar) only one possible

$\mathfrak{gl}(F_3) \times \mathfrak{gl}(F_1)$ map of free $U(\mathbb{L})$-modules in each case, so both differentials have to be the same.

Thus the complexes $\mathcal{K}^*(\sigma, \tau, t)$ are exact at the middle term so Corollary 17.2 assures that the complex $\mathbb{F}^{gen}_\bullet$ is acyclic over $\hat{R}_{gen}$.

Let us prove that the pair $(\hat{R}_{gen}, \mathbb{F}^{gen}_\bullet)$ has the universality property. It was constructed by killing a series of cycles in the Koszul complex of $I(d_3)$. In every realization $(S, \mathbb{G}_\bullet)$ where S is Noetherian and $\mathbb{G}_\bullet$ is a resolution of format (r_1, r_2, r_3) these cycles are boundaries. This and the universal property of the ideal transform give a homomorphism $\phi : \hat{R}_{gen} \to S$ such that $\mathbb{G}_\bullet = \mathbb{F}^{gen}_\bullet{}_\bullet \otimes_{\hat{R}_{gen}} S$.

This completes the proof that $\hat{R}_{gen}$ is indeed a generic ring.

To prove the part about the deformation, let us decompose the ring $\hat{R}_{gen}$ to the $\mathfrak{gl}(F_2) \times \mathfrak{gl}(F_0)$ isotypic components.

$$\hat{R}_{gen} = \oplus_\mu \hat{R}_{gen,\mu} = \oplus_\mu S_{\phi(\mu)} F_0 \otimes S_{\theta(\mu)} F_2 \otimes V_{\lambda(\sigma(\mu), \tau(\mu), \boldsymbol{a})}$$

The cokernel of the complex $\mathcal{K}^*(\sigma, \tau, t)$ (i.e. the parabolic BGG complex) is an irreducible highest weight module for $\mathfrak{g}(T_{p,q,r})$, so the component $\hat{R}_{gen,\mu}$ acquires the structure of an irreducible $\mathfrak{g}(T_{p,q,r}) \times \mathfrak{gl}(F_2) \times \mathfrak{gl}(F_0)$ lowest weight module. This action on $\oplus_\mu \hat{R}_{gen,\mu}$ is obviously multiplicity free. The problem is that this does not give the structure of the $\mathfrak{g}(T_{p,q,r}) \times \mathfrak{gl}(F_2) \times \mathfrak{gl}(F_0)$ module on the ring $\hat{R}_{gen}$ because the multiplication might not be $\mathfrak{g}(T_{p,q,r}) \times \mathfrak{gl}(F_2) \times \mathfrak{gl}(F_0)$ equivariant.

However for every two pieces $\hat{R}_{gen,\mu}$ and $\hat{R}_{gen,\nu}$ their product goes to the sum of several graded pieces with the extremal one being $\hat{R}_{gen,\mu+\nu}$. We can deform the multiplication on $\hat{R}_{gen}$ by shrinking the other components of the product to zero. This gives us a new commutative algebra

$$\hat{R}_{spec} := \oplus_{\mu \in \Lambda} \hat{R}_{gen,\mu}.$$

The connection between the rings $\hat{R}_{gen}$ and $\hat{R}_{spec}$ was explained in Grosshans lecture notes [21], chapter 15. In theorem 15.14 Grosshans showed that there is an algebra D which is a free $\mathbb{C}[x]$ module such that the general fibre of the resulting map

$$\pi : Spec\ D \to \mathbb{C}$$

over a point $z \in \mathbb{C}$ is isomorphic to $Spec\ \hat{R}_{gen}$ and the fibre over 0 is isomorphic to $\hat{R}_{spec}$. The next point is that the Cartan part of the multiplication map

$$\hat{R}_{gen,\mu} \otimes \hat{R}_{gen,\nu} \to \hat{R}_{gen,\mu+\nu}$$

is not only $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}_+(T_{p,q,r})$-equivariant, but also $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}(T_{p,q,r})$-equivariant (so an epimorphism). The reason is as follows. It is well-known (see [33], chapter X) that the homogeneous coordinate ring of the homogeneous space $\mathcal{G}/\mathcal{P}$ is a direct sum of irreducible representations $\oplus_{\lambda \in \Lambda} V(\lambda)$ of irreducible representations $V(\lambda)$ of $\mathfrak{g}(T_{p,q,r})$ with Λ consisting of all weights of type $\lambda(\sigma(\mu), \tau(\mu), \boldsymbol{a})$. The multiplication map in this ring is just the Cartan multiplication $V(\lambda_1) \otimes V(\lambda_2) \to V(\lambda_1 + \lambda_2)$, which is an epimorphism.

In order to compare the multiplications in the homogeneous coordinate ring of $\mathcal{G}/\mathcal{P}$ and the map $\hat{R}_{gen,\mu} \otimes \hat{R}_{gen,\nu} \to \hat{R}_{gen,\mu+\nu}$ we need one more fact.

The complexes $\mathcal{K}(\sigma,\tau,t)$ we got in [47] as isotypic components of the spectral sequence, before dualizing to get $\mathcal{K}^*(\sigma,\tau,t)$ have another interpretation in terms of Grothendieck-Cousin complex introduced by George Kempf in [30]. The precise definitions of all the notions using in the remainder of this section can be found in [33], chapter 9.

The terms of the Grothendieck-Cousin complex are the local cohomology modules associated to the stratification of the homogeneous space $Z := \mathcal{G}/\mathcal{P}$ where $\mathcal{G}$ is the Kac-Moody group corresponding to $T_{p,q,r}$ and $\mathcal{P}$ is the parabolic subgroup corresponding to the simple root corresponding to the vertex z_1 (see section 3). The homogeneous space Z has a stratification by Schubert cells and we denote by Z_i the closed subset which is a union of all Schubert cells of codimension $\geq i$. Let $\mathcal{V}(\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})$ be a homogeneous vector bundle on Z corresponding to the weight $\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})$.

Proposition 18.2. *([30], [33], section 9.2) The isotypic component $\mathcal{K}(\sigma,\tau,t)$ of the spectral sequence is the beginning part of the Grothendieck-Cousin complex*

$$0\to H^0_{Z_0/Z_1}(Z,\mathcal{V}(\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a}))\to H^1_{Z_1/Z_2}(Z,\mathcal{V}(\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a}))\to$$
$$\to H^2_{Z_2/Z_3}(Z,\mathcal{V}(\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})).$$

Now, looking at two weights $(\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})$ and $(\lambda(\sigma(\nu),\tau(\nu),\boldsymbol{a}')$ we see that both the Cartan part of their multiplication in $\hat{R}_{gen}$ and the multiplication of the elements of the kernels of Grothendieck-Cousin complexes from Proposition 18.2 are the same because they come from multiplication of sections on the open set U_{gen} which, as we noted in Remark 16.4, is just the open Schubert cell, i.e. $Z_0\setminus Z_1$.

This allows us to prove that if $T_{p,q,r}$ is a Dynkin diagram, then the rings $\hat{R}_{gen}$ and $\hat{R}_{spec}$ are Noetherian.

We know that the Lie algebra $\mathbb{L}(r_1+1,F_3,F_1)$ is finite dimensional if and only if $T_{p,q,r}$ is a Dynkin diagram. In such a case all irreducible highest weight modules for $\mathfrak{g}(T_{p,q,r})$ are finite dimensional. Therefore it is enough to show that the semigroup of weights occurring in $\hat{R}_{gen}$ is finitely generated. But this semigroup is the semigroup of the terms in our spectral sequence which give the contribution to H_0. Thus we get the set of sextuples $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\alpha,\beta,\gamma)$ with $\boldsymbol{a}\in\mathbb{Z}$, $\boldsymbol{b},\boldsymbol{c}\in\mathbb{N}$ such that all the weights

$$(\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_1,\ldots,\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}+\alpha_{r_3-1},\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}),$$
$$(\boldsymbol{b}-\boldsymbol{c}+\beta_1,\ldots,\boldsymbol{b}-\boldsymbol{c}+\beta_{r_2-1},\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c},-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_{r_3-1},\ldots,-\boldsymbol{a}+\boldsymbol{b}-\boldsymbol{c}-\alpha_1)$$
$$(\boldsymbol{c}+\gamma_1,\ldots,\boldsymbol{c}+\gamma_{r_1-1},\boldsymbol{c},\boldsymbol{c}-\boldsymbol{b},\boldsymbol{c}-\boldsymbol{b}-\beta_{r_2-1},\ldots,\boldsymbol{c}-\boldsymbol{b}-\beta_1),$$
$$(0^{f_0-r_1},-\boldsymbol{c},-\boldsymbol{c}-\gamma_{r_1-1},\ldots,-\boldsymbol{c}-\gamma_1)$$

are dominant. This translates to the condition that $\boldsymbol{a}\geq 0$, so our semigroup is finitely generated.

□

Let us summarize the properties of $\hat{R}_{spec}$.

Proposition 18.3. *We have an $\mathfrak{sl}(F_0)\times\mathfrak{sl}(F_2)\times\mathfrak{g}(T_{p,q,r})$ decomposition*

$$\hat{R}_{spec}=\oplus_\mu S_{\phi(\mu)}F_0\otimes S_{\theta(\mu)}F_2\otimes V_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$$

where V_λ is the irreducible lowest weight module of weight λ for $\mathfrak{g}(T_{p,q,r})$. It is the highest weight representation for the opposite Borel subalgebra. It is also irreducible. The ring $\hat{R}_{spec}$ is a multiplicity free representation of $\mathfrak{sl}(F_0) \times \mathfrak{sl}(F_2) \times \mathfrak{g}(T_{p,q,r})$. Its lattice of weights is saturated.

Remark 18.4.
1. The easiest way to identify the module $V_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$ is as follows. This module has a grading

$$V_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})} = \oplus_{i\geq 0} V^{(i)}_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$$

induced by the grading on $\mathfrak{g}(T_{p,q,r})$. The lowest graded component $V^{(0)}_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$ is the representation of $GL(F_3) \times GL(F_1)$ that occurs in the $GL(F_3) \times GL(F_1)$ isotypic component of R_a corresponding to $(\sigma(\mu), \tau(\mu), \boldsymbol{a})$. This identification allows also to describe the correct $GL(F_3) \times GL(F_1)$ structure of higher graded components of $V_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$. The multiplication by the first component $F_3^* \otimes \bigwedge^{r_1+1} F_1$

$$F_3^* \otimes \bigwedge^{r_1+1} F_1 \otimes V^{(i)}_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})} \to V^{(i+1)}_{\lambda(\sigma(\mu),\tau(\mu),\boldsymbol{a})}$$

has to be $GL(F_3) \times GL(F_1)$-equivariant.
2.I expect that the rings $\hat{R}_{gen}$ are also $\mathfrak{gl}(F_0) \times \mathfrak{gl}(F_2) \times \mathfrak{g}(T_{p.q.r})$ equivariant. In fact it is enough to check the quadratic relations more precisely to see that they really hold in $\hat{R}_{gen}$. In every example analyzed below it is true.

Let us exhibit the decomposition of $\hat{R}_{spec}$ explicitly. For given α, β, t we define the weight $\lambda(\sigma, \tau, t)$ of $\mathfrak{g}(T_{p,q,r})$ as follows. We label the vertices of $T_{p,q,r}$ on the third arm by the coefficents of fundamental weights in σ, i.e.

$$\lambda_{p+q+i} = \sigma_{r-1-i} - \sigma_{r-i}$$

for $i = 1, \dots, r-2$. We also label the vertices at the center and the first two arms by coefficients of fundamental weights in τ, i.e.

$$\lambda_0 = \tau_p - \tau_{p+1},$$

$$\lambda_i = \tau_{p-i} - \tau_{p-i+1}$$

for $1 \leq i \leq p-1$, and

$$\lambda_i = \tau_i - \tau_{i+1}$$

for $i = p+1, \dots, p+q-1$. Finally, we put

$$\lambda_{p+q} = \boldsymbol{a}.$$

We also set $t := \boldsymbol{a} + 1$.

For a sextuple $\mu = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \alpha, \beta, \gamma)$ with $\boldsymbol{a} \geq 0$, as in the decomposition of $(q_a)_* \mathbb{F}^a_\bullet$ we define

$$\sigma(\mu) := (\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c} + \alpha_1, \dots, \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c} + \alpha_{r_3-1}, \boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}),$$

$$\tau(\mu) := (\boldsymbol{c} + \gamma_1, \dots, \boldsymbol{c} + \gamma_{r_1-1}, \boldsymbol{c}, \boldsymbol{c} - \boldsymbol{b}, \boldsymbol{c} - \boldsymbol{b} - \beta_{r_2-1}, \dots, \boldsymbol{c} - \boldsymbol{b} - \beta_1),$$

$$\theta(\mu) :=$$

$$= (\boldsymbol{b} - \boldsymbol{c} + \beta_1, \dots, \boldsymbol{b} - \boldsymbol{c} + \beta_{r_2-1}, \boldsymbol{b} - \boldsymbol{c}, -\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}, -\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} - \alpha_{r_3-1}, \dots, -\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c} - \alpha_1),$$

$$\phi(\mu) := (0^{f_0 - r_1}, -\boldsymbol{c}, -\boldsymbol{c} - \gamma_{r_1-1}, \dots, -\boldsymbol{c} - \gamma_1).$$

For the formats for which the algebra is not finite dimensional we do not have a Noetherian generic ring $\hat{R}_{gen}$. Still the multiplicity free structure of $\hat{R}_{spec}$ could be useful for applications.

Remark 18.5. *In particular we proved the conjecture from* [40] *stating that the generic ring* $\hat{R}_{gen}$ *constructed there for the format* $(f_0, f_1, f_2, f_3) = (1, n, n, 1)$ *is Noetherian. However the generators stated in* [40] *in that case are not correct. See the next section for more precise analysis of this case.*

19. The case $n = 3$; first applications.

In previous sections for each format (f_0, f_1, f_2, f_3) the specific generic ring $\hat{R}_{gen}$ was constructed from R_a by a procedure of killing cycles. The ring $\hat{R}_{gen}$ is Noetherian only in very few cases. Let us recall that the combinatorics of ranks works as follows. To three ranks (r_1, r_2, r_3) we associate the triple

$$(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$$

and we look at the graph $T_{p,q,r}$

$$\begin{array}{ccccccccccccc}
x_{p-1} & - & x_{p-2} & \dots & x_1 & - & u & - & y_1 & \dots & y_{q-2} & - & y_{q-1} \\
 & & & & & & | & & & & & & \\
 & & & & & & z_1 & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & \dots & & & & & & \\
 & & & & & & z_{r-2} & & & & & & \\
 & & & & & & | & & & & & & \\
 & & & & & & z_{r-1} & & & & & &
\end{array}$$

The ring $\hat{R}_{gen}$ is Noetherian if and only if the graph $T_{p,q,r}$ is a Dynkin graph.

For the resolutions of cyclic modules (i.e. those with $r_1 = 1$) this means that $T_{p,q,r}$ has to be one of the following.

- Cases A_n $(n \geq 3)$, i.e. triples $(p, q, r) = (2, 1, n - 1)$,
- Cases D_n $(n \geq 4)$, i.e. triples $(p, q, r) = (2, n - 2, 2)$ and $(p, q, r) = (2, 2, n - 2)$,
- Case E_6, i.e. a triple $(p, q, r) = (2, 3, 3)$,
- Cases E_7, i.e. triples $(p, q, r) = (2, 3, 4)$ and $(p, q, r) = (2, 4, 3)$,
- Cases E_8, i.e. triples $(p, q, r) = (2, 3, 5)$ and $(p, q, r) = (2, 5, 3)$.

Thus it is an important problem to describe the rings $\hat{R}_{gen}$ in these cases as explicitly as possible. Such description would allow to "map" these resolutions as different generators could be considered to be coordinates in the variety parametrizing such resolutions.

The rings $\hat{R}_{gen}$ have remarkable properties. They have ([49], Proposition 9.3) a decomposition analogous to R_a. We will write the decomposition for R_a given above in a more compact way,

$$R_a = \oplus_{\lambda \in \Lambda} S_{\alpha(\lambda)} F_3 \otimes S_{\beta(\lambda)} F_2 \otimes S_{\gamma(\lambda)} F_1 \otimes S_{\delta(\lambda)} F_0$$

where Λ is some lattice of highest weights, then

$$\hat{R}_{gen} = \oplus_{\lambda \in \Lambda} S_{\beta(\lambda)} F_2 \otimes S_{\delta(\lambda)} F_0 \otimes V(\alpha(\lambda), \gamma(\lambda))$$

where $V(\theta)$ is certain lowest weight module for the Kac-Moody Lie algebra corresponding to the diagram $T_{p,q,r}$.

Recall that the connection of the graph $T_{p,q,r}$ with our free resolution is as follows. After removing the vertex z_1 we get the graph $T_{p,q,r}$ with two connected components. The component containing vertices x_i, u, y_j $(1 \le i \le p-1, 1 \le j \le q-1)$ is thought of as the Dynkin diagram of the root system of F_1 and the component containing vertices z_k $(2 \le k \le r-1)$ is thought of as the Dynkin diagram of the root system of F_3. The nice thing that happens is that $\hat{R}_{gen}$ has bigger symmetry expressed by the action of the Lie algebra corresponding to the graph $T_{p,q,r}$.

We will study more precisely the generators of the generic ring $\hat{R}_{gen}$. The set of generators described in section 18 is too big, since it uses a deformation to $\hat{R}_{spec}$. In fact we will see that much smaller set in fact generates $\hat{R}_{gen}$. Obviously the ring R_a is generated by representations containing entries of three differentials d_3, d_2, d_1 and the Buchsbaum-Eisenbud multipliers a_3, a_2, a_1. The corresponding isotypic components of $\hat{R}_{gen}$ are the natural candidates for the generators.

Let us restrict to resolutions of cyclic modules, i.e, the formats with $r_1 = 1$. We identify three critical representations of $\hat{R}_{gen}$. These are isotypic components

$$W(d_3) = F_2^* \otimes V(\omega_{z_{r-1}}),$$

$$W(d_2) = F_2 \otimes V(\omega_{y_{q-1}}),$$

$$W(a_2) = \mathbb{C} \otimes V(\omega_{x_1}).$$

For cyclic formats these are enough, as a_3 obviously consists of minors of d_3, and a_1 is just one element (additional variable) which is the greatest common divisor of entries of d_1. We will disregard it in the future. The differential d_1 is a_1 multiplied by a_2, so it is also redundant.

We will pay special attention to the graded components of these critical representations, as these contain the most important structure theorems, providing, in a way the coordinates of the corresponding moduli space.

We also introduce more systematic notation regarding various structure theorems. The i-th graded component of $W(d_3)$ is denoted $v_i^{(3)}$. The j-th graded component of $W(d_2)$ is denoted $v_j^{(2)}$. The k-th graded component of $W(a_2)$ is denoted $v_k^{(1)}$. We use the convention that the lowest graded component occurs in degree 0, so $v_0^{(3)} = d_3$, $v_0^{(2)} = d_2$, $v_0^{(1)} = a_2$.

We will note (it will become more clear later) that for the formats of cyclic modules in fact the representation $W(a_2)$ is generated by $W(d_3)$ and $W(d_2)$, as the Leibniz formula for multiplying elements of F_1 and F_2 will show.

Besides studying the structure of the generic ring $\hat{R}_{gen}$ it is important to study the examples of these resolutions occurring in algebraic geometry in order to understand how their structure might help in understanding these examples.

In these notes we try to reduce representation theory of exceptional Lie algebras or Kac-Moody Lie algebras to a minimum, so we will just state the facts we need.

One of the main conjectures ([15]) we made whose aim would be to show that Dynkin formats are indeed special is the following

Conjecture 19.1. *The Dynkin formats are precisely the formats such that any perfect ideal I in a regular local ring R, such that the minimal free resolution of R/I has a Dynkin format is in the linkage class of a complete intersection.*

We will refer to this conjecture as a LICCI Conjecture. One can consult [15] for the examples based on Macaulay inverse systems showing that for non-Dynkin format we cannot hope it satisfies the condition of LICCI conjecture.

20. Graded cases.

Let us make some remarks about the graded cases. Consider a graded (by natural numbers) ring S and a free resolution $\mathbb{G}_\bullet$ of a graded module of format (f_0, f_1, f_2, f_3) over S. This means we can assign the degrees to all basis vectors of G_0, G_1, G_2, G_3 so the differentials in $\mathbb{G}_\bullet$ have degree zero.

Proposition 20.1. *There is a canonical choice of grading on $\hat{R}_{gen}$ such that the homomorphism $\phi : \hat{R}_{gen} \to S$ can be chosen to be homogeneous of degree zero.*

Proof. The grading in question is constructed as follows. We assign the degrees to basis vectors of F_0, F_1, F_2, F_3 to be the same as the degrees of corresponding basis elements in $G_0, G_{1,2}, G_3$. This means (in terms of representation theory) that there are degrees assigned to basis vectors u_m, e_i, f_j, g_k. Denote $deg(u_m) = \delta_m$, $deg(e_i) = \epsilon_i$, $deg(f_j) = \phi_j$, $deg(g_k) = \psi_k$. Then we can assign the degree to any tensor in $\hat{R}_{gen}$ by adding the degrees of all its tensor components, keeping in mind that the degree of the dual vector is the negative of the degree of the vector. For example, the degree of the tensor $g_k^* \otimes f_j^* \otimes e_i$ is $-\psi_k - \phi_j + \epsilon_i$.

Then it is clear that the homomorphism ϕ can be chosen to be graded of degree zero, just by choosing all liftings to be homogeneous. □

There is an important notion related to the graded situation.

Definition 20.2. *Let us consider the pair $(S, \mathbb{G}_\bullet)$ as above, and let us choose the grading on $\hat{R}_{gen}$ so the homomorphism ϕ is homogeneous of degree zero. Let N ba a natural number. The complex $\mathbb{G}_\bullet$ is* **of order** *N if for each of the critical representations $W(d_3)$, $W(d_2)$, $W(a_2)$ and for each natural number M bigger that N the M-th graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ are sent to 0 by ϕ. In particular this occurs when the degrees of tensors in the M-th graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ are negative. The complex $\mathbb{G}_\bullet$ is* **of finite order** *if there exists an N such that $\mathbb{G}_\bullet$ is of order N.*

There seems to be a close link between the finite order notion and LICCI ideals.

Assume we deal with the format $(1, f_1, f_2, f_3)$. Let $(S, \mathbb{G}_\bullet)$ be a pair where S is graded and $\mathbb{G}_\bullet$. resolves the S-module S/I where I is perfect of codimension 3. Note that to go from a component M to $M + 1$ of any critical representation we multiply by a tensor from $F_3^* \otimes \bigwedge^2 F_1$.

Remark 20.3. (1) *In* [26] *Huneke and Ulrich prove that if we have*

$$max_k\ \psi_k \leq 2min_i\ \epsilon_i$$

then the ideal I cannot be LICCI. Note that if this is true, then every element in $F_3^* \otimes \bigwedge^2 F_1$ has a positive degree. This means the degrees of tensors in graded components of critical representations do not go down, so they are unlikely to be zero, i.e. the resolution $\mathbb{G}_\bullet$ should have an infinite order.

(2) Similarly, in [27] *Huneke and Ulrich prove that if we have inequality*

$$min_k\ \psi_k > 2max_i\ \epsilon_i$$

then they expect that the ideal will be LICCI. This condition, however, means that in all critical representations the degrees of tensors in the M-th graded component go strictly down, so they have to eventually become negative, so the resolution $\mathbb{G}_\bullet$ will have finite order.

(3) However the situation is likely to be more complicated than "finite order means LICCI" as the examples in [25] *indicate. It would be very interesting to check these examples in detail.*

(4) It would be very interesting to investigate the order of the n-th generic link of a complete intersection. One would need to investigate different possible gradings on this ring and the the degrees of tensors of N-th graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ for each choice.

Example 20.4. *Let us deal with the smallest non-Dynkin format $(1, 6, 8, 3)$. Consider the pair $(S, \mathbb{G}_\bullet)$ which is either an Eagon-Northcott complex of 2×2 minors of a 2×4 matrix, or a resolution of 2×2 minors of a 3×3 symmetric matrix. Then $\mathbb{G}_\bullet$ is not of finite order.*

Proof. We have $deg(u_1) = 0$, $deg(e_i) = 2$, $deg(f_j) = 3$, $deg(g)_k = 4$. Then it is easy to see by degree count that for each M the tensors in the M-th graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ have degree 1 and one can show that they can be chosen to be non-zero. □

21. Example: formats $(1, n, n, 1)$.

Let us start with the simplest formats $(1, n, n, 1)$.

In this format we deal with the special orthogonal Lie algebra $\underline{so}(U)$ where U is the orthogonal space

$$U = F_1^* \oplus F_1$$

with the quadratic form which is a duality pairing on $F_1^* \oplus F_1$. The grading on this Lie algebra is

$$\underline{so}(U) = \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1$$

where $\underline{g}_0 = \underline{sl}(F_1) \oplus \mathbf{C} = \underline{gl}(F_1)$, $\underline{g}_1 = \bigwedge^2 F_1$, $\underline{g}_{-1} = \underline{g}_1^*$.

The generic ring is obtained from the ring R_a in one stage, by lifting the cycle giving Koszul relations

$$\begin{tikzcd} 0 \arrow[r] & F_3 \arrow[r, "d_3"] & F_2 \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] & F_0 \\ & & & \bigwedge^2 F_1 \arrow[ul, "v_1^{(3)}"] \arrow[u, "r_1^{(3)}"'] & \end{tikzcd}$$

which gives the defect $F_3^* \otimes \bigwedge^2 F_1$.

It was proved already in [40] that in this way we get a generic ring. This was done by producing a family of complexes over $Sym(\bigwedge^2 F_1)$ resolving certain family of modules, without noticing that these are parabolic BGG resolutions. One has to mention that at that time the parabolic BGG resolutions were not yet invented.

Let us describe the critical representations in $\hat{R}_{gen}$. They are the $\underline{gl}(F_2)\times\underline{gl}(F_0)$-equivariant components of $\hat{R}_{gen}$ containing the tensors corresponding to d_3, d_2 and a_2, i.e. the components of F_2^*, F_2 and $\mathbf{C}$ respectively. We have

$$W(d_3) = F_2^* \otimes [\oplus_{k\geq 0} S_{1-k}F_3 \otimes \bigwedge^{2k} F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1],$$

$$W(a_2) = \mathbf{C} \otimes [\oplus_{k\geq 0} S_k F_3^* \otimes \bigwedge^{k+1} F_1].$$

Recall that we denote the tensors in the graded components of three critical representations $W(d_3)$, $W(d_2)$, $W(a_2)$ by $v_i^{(3)}$, $v_j^{(2)}$, $v_k^{(1)}$ respectively. This explains the notation above.

These representations (actually $W(d_3)$, $W(d_2)$ suffice) give generators of the generic ring $\hat{R}_{gen}$. Let us make a few basic observations. Three representations described above acquire the grading induced by the grading on $\underline{so}(U)$. In the lowest degree we get just the representations $v_0^{(3)} = d_3$, $v_0^{(2)} = d_2$ and $v_0^{(1)} = a_2$ from R_a. This is a general phenomenon, for each irreducible representation in $\hat{R}_{gen}$ its lowest degree term will just give the corresponding representation from R_a. Looking at the next degree term in our three representations we see the tensors $F_2^*\otimes\bigwedge^2 F_1$ $(v_1^{(3)})$, $F_2\otimes F_3^*\otimes F_1$ $(v_1^{(2)})$ and $F_3^*\otimes\bigwedge^3 F_1$ $(v_1^{(1)})$. These tensors can be thought of as maps $\bigwedge^2 F_1 \to F_2$, $F_1\otimes F_2\to F_3$ and $\bigwedge^3 F_1\to F_3$. These are the components of the multiplicative structure on $\mathbf{F}_\bullet$. Why do we know that ? It was already proved by Buchsbaum and Eisenbud that every finite free resolution of length three has an associative, graded commutative algebra structure. So the components of this structure have to sit in the generic ring. But one can easily see that these three representations occur in $\hat{R}_{gen}$ only once, so they have to be the tensors giving the multiplicative structure.

The other components of three representations $W(d_3)$, $W(d_2)$, $W(a_2)$ also have similar interpretation in terms of the resolution. Let us look at some resolution

$$0\to G_3 \xrightarrow{d_3} G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} R$$

of format $(1,n,n,1)$ over some Noetherian commutative ring R. We have a comparison map from the Koszul complex on d_1 to our resolution

$$\begin{array}{ccccccccc}
0 & \to & G_3 & \xrightarrow{d_3} & G_2 & \xrightarrow{d_2} & G_1 & \xrightarrow{d_1} & R \\
 & & \uparrow\ \alpha_3 & & \uparrow\ \alpha_2 & & \uparrow\ \alpha_1 & & \uparrow\ \alpha_0. \\
\bigwedge^4 G_1 & \xrightarrow{\delta_4} & \bigwedge^3 G_1 & \xrightarrow{\delta_3} & \bigwedge^2 G_1 & \xrightarrow{\delta_2} & G_1 & \xrightarrow{d_1} & R
\end{array}$$

The maps $\alpha_2 = v_1^{(3)}$, $\alpha_3 = v_1^{(1)}$ give us the first graded components of $W(d_3)$ and $W(a_2)$.

Now, the composition $d_3\alpha_3\delta_4 = 0$, but d_3 is injective, so $\alpha_3\delta_4 = 0$. But the entries of the matrix δ_4 involves only the four generators of the resolved ideal, so the last equation can be rewritten as such relation. This can be converted to the claim that we have a complex

$$\bigwedge^5 G_1 \xrightarrow{\delta_5} \bigwedge^4 G_1 \xrightarrow{\alpha_3} G_1 \xrightarrow{d_1} R,$$

where by abuse of notation we denote by α_3 the contraction by α_3.

So we have a comparison map

$$\begin{array}{ccccccccc} 0 & \to & G_3 & \stackrel{d_3}{\to} & G_2 & \stackrel{d_2}{\to} & G_1 & \stackrel{d_1}{\to} & R \\ & & \uparrow\ \beta_3 & & \uparrow\ \beta_2 & & \uparrow\ \alpha_1 & & \uparrow\ \alpha_0. \\ \bigwedge^6 G_1 & \stackrel{\delta_6}{\to} & \bigwedge^5 G_1 & \stackrel{\alpha_3}{\to} & \bigwedge^2 G_1 & \stackrel{\delta_2}{\to} & G_1 & \stackrel{d_1}{\to} & R \end{array}$$

The maps $\beta_2 = v_2^{(3)}$, $\beta_3 = v_2^{(1)}$ give us the second graded components of of $W(d_3)$ and $W(a_2)$. Now, the composition $d_3\beta_3\delta_6 = 0$, but d_3 is injective, so $\beta_3\delta_6 = 0$. But the entries of the matrix δ_6 involves only the four generators of the resolved ideal, so the last equation can be rewritten as such relation. This can be converted to the claim that we have a complex

$$\bigwedge^7 G_1 \stackrel{\delta_7}{\to} \bigwedge^6 G_1 \stackrel{\beta_3}{\to} G_1 \stackrel{d_1}{\to} R,$$

where by abuse of notation we denote by β_3 the contraction by β_3, and we continue like that to get all higher components of $W(d_3)$ and $W(a_2)$.

Again we know that the representations giving higher components of $W(d_3)$ and $W(a_2)$ have to specialize to the factorizations we just constructed because they are the only representations of this type occurring in $\hat{R}_{gen}$.

For other Dynkin formats it would be desirable to give similar interpretations.

Next we turn to another problem. We want to look at the perfect ideals of our format. One of the ways to get them is to look at the open set

$$U_{CM} \subset Spec(\hat{R}_{gen})$$

consisting of points where the complex $(\mathbf{F}_\bullet^{gen})^*$ is acyclic.

The format $(1, n, n, 1)$ is helpful because we know the answer by Buchsbaum-Eisenbud Theorem: perfect ideals with a resolution of format $(1, n, n, 1)$ exist if and only if n is odd and they are given by Pfaffians of an odd-sized skew-symmetric matrix. So let us look at the format $(1, n, n, 1)$ with n odd. One can recall that the key point in Buchsbaum-Eisenbud Theorem is that for a resolution of a perfect ideal of format $(1, n, n, 1)$ the multiplication

$$F_1 \otimes F_2 \to F_3 = R$$

gives a perfect pairing of F_1 and F_2. So this means that the entries of that tensor do not lie in a maximal ideal of R (assuming our resolution is over a local ring R).

But this multiplication is the tensor in the top graded component of $W(d_2)$.

This gives an idea of looking at the top graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$.

Let us look at the "forbidden" format $(1, n, n, 1)$, n even. In this case we see that the three top components of our representations are: $F_2^* \otimes F_3^*$, $F_2 \otimes F_1$ and F_1^*. These three tensors can be thought of as three maps

$$F_3^* \to F_2 \to F_1^* \to R$$

over $R = \hat{R}_{gen}$. Moreover, it is easily seen that these tensors give us a complex, as there are no representations in $\hat{R}_{gen}$ giving compositions of these maps.

This also brings out the key difference between the formats $(1, n, n, 1)$ for n even and odd. For n even two half-spinor representations are selfdual, for n odd they are dual to each other. It is natural to conjecture the following (this statement was first announced in [14]).

Conjecture 21.1. *The point from $Spec(\hat{R}_{gen})$ is in U_{CM} if and only if the tensors giving top graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ give a split exact complex.*

Let us denote U_{split} the open set of points in $Spec(\hat{R}_{gen})$ where the tensors giving top graded components of $W(d_3)$, $W(d_2)$, $W(a_2)$ give a split exact complex.

This gives the following idea. Since the ring $\hat{R}_{gen}$ has an action of our orthogonal Lie algebra, in order to see a generic point of U_{split} we can use the involution of $\underline{so}(U)$ interchanging highest and lowest weights, and then it is enough to calculate all the factorizations giving components of $W(d_3)$, $W(d_2)$, $W(a_2)$ using the defect variables. We should get a nice resolution of a perfect ideal.

Let us see what happens for the smallest format $(1,4,4,1)$. We take the split exact complex

$$0 \to R \xrightarrow{d_3} R^4 \xrightarrow{d_2} R^4 \xrightarrow{d_1} R.$$

Let $\{e_1, e_2, e_3, e_4\}$ will be the basis of F_1, $\{f_1, f_2, f_3, f_4\}$ will be the basis of F_2, $\{g\}$ will be the basis of F_3, such that the differentials in these bases are given by matrices

$$d_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, d_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, d_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix},$$

Calculating the multiplicative structure we get

$$e_i \cdot e_j = b_{ij} f_4,$$

for $1 \le i, j \le 3$,

$$e_i \cdot e_4 = -f_i + b_{i4} f_4,$$

for $1 \le i \le 3$.

Let us calculate the tensor giving the component $v_2^{(3)}$. This is a factorization

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^2 F_3 \arrow[r, "d_3"] & F_3 \otimes F_2 \arrow[r, "d_3"] & S_2 F_2 \arrow[r, "S_2(d_2)"] & S_2 F_1 \\ & & & \bigwedge^4 F_1 \arrow[u, "r_2^{(3)}"'] \arrow[ul, "v_2^{(3)}"] & \end{tikzcd}$$

with $r_2^{(3)} = S_2(p_1)$. We get that

$$v_2^{(3)}(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = b_{12} g \otimes (e_3 \cdot e_4) - b_{13} g \otimes (e_2 \cdot e_4) + b_{14} g \otimes (e_1 \cdot e_4)$$

which, written as a matrix, gives

$$v_2^{(3)} = \begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \\ Pf((b_{ij})) \end{pmatrix}.$$

Note that after row operations we just get the matrix

$$\begin{pmatrix} -b_{23} \\ b_{13} \\ -b_{12} \\ 0 \end{pmatrix}.$$

Lifting similarly the other maps we get that (after applying the row and column operations)

$$v_1^{(2)} = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ -b_{12} & 0 & b_{23} & 0 \\ -b_{13} & -b_{23} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

ans

$$v_2^{(1)} = \begin{pmatrix} b_{23} & b_{13} & b_{12} & 0 \end{pmatrix}.$$

This means that the complex $\mathbf{F}_\bullet^{top}$ is a generic resolution of format $(1,3,3,1)$ plus a splitting factor $R \xrightarrow{1} R$. Similarly (it is the easiest to lift $v_1^{(2)}$) we see that for the format $(1,n,n,1)$, n even we get in the same way that the complex $\mathbf{F}_\bullet^{top}$ is a direct sum of Buchsbaum-Eisenbud generic resolution of a perfect ideal of format $(1,n-1,n-1,1)$ (given by submaximal Pfaffians of odd-sized skew-symmetric matrix) and a splitting factor $R \xrightarrow{1} R$. This actually proves Conjecture 21.1 for the format $(1,n,n,1)$ with n even.

22. Example: formats $(1,4,n,n-3)$.

In this format we deal with the special orthogonal Lie algebra which can be identified with the orthogonal space

$$U = F_3^* \oplus \bigwedge^2 F_1 \oplus F_3$$

with the quadratic form which is a duality pairing on $F_3^* \oplus F_3$ and the exterior multiplication on $\bigwedge^2 F_1$. The grading on this Lie algebra is

$$\underline{so}(U) = \underline{g}_{-2} \oplus \underline{g}_{-1} \oplus \underline{g}_0 \oplus \underline{g}_1 \oplus \underline{g}_2$$

where $\underline{g}_0 = \underline{sl}(F_3) \oplus \underline{sl}(F_1) \oplus \mathbf{C}$, $\underline{g}_1 = F_3^* \otimes \bigwedge^2 F_1$, $\underline{g}_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1$, $\underline{g}_{-i} = \underline{g}_i^*$.

The generic ring is obtained from the ring R_a in two stages. First, we lift the cycle giving Koszul relations

$$\begin{array}{ccccccccc} 0 & \to & F_3 & \xrightarrow{d_3} & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 \\ & & & & & v_1^{(3)} \nwarrow & \uparrow\ r_1^{(3)} & & \\ & & & & & & \bigwedge^2 F_1 & & \end{array}$$

which gives the defect $F_3^* \otimes \bigwedge^2 F_1$ and then we kill the cycle

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^2 F_3 & \xrightarrow{d_3} & F_3 \otimes F_2 & \xrightarrow{d_3} & S_2 F_2 & \xrightarrow{S_2(d_2)} & S_2 F_1 \\ & & & & & v_2^{(3)} \nwarrow & \uparrow\ r_2^{(3)} & & \\ & & & & & & \bigwedge^4 F_1 & & \end{array}$$

where $r_2^{(3)} = S_2(v_1^{(3)})$, with defect $\bigwedge^2 F_3^* \otimes \bigwedge^4 F_1$.

Three critical representations in $\hat{R}_{gen}$ have the following decompositions.

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \otimes \bigwedge^{even} F_3^* \oplus F_1 \otimes \bigwedge^{odd} F_3^*],$$

$$W(a_2) = \mathbf{C} \otimes [F_1 \otimes \bigwedge^{even} F_3^* \oplus F_1^* \otimes \bigwedge^{odd} F_3^*].$$

Notice the three top components of these three representations. In $dimF_3 = 2m$ is even they are: $F_2^* \otimes F_3^*$, $F_2 \otimes F_1^*$ and F_1. These three can be thought of as three maps

$$F_3^* \otimes \hat{R}_{gen} \overset{d_3^{top}}{\to} F_2 \otimes \hat{R}_{gen} \overset{d_2^{top}}{\to} F_1 \otimes \hat{R}_{gen} \overset{d_1^{top}}{\to} \hat{R}_{gen}.$$

It is not difficult to see they form a complex over $\hat{R}_{gen}$. Similarly, if $dimF_3 = 2m+1$ is odd, the three top graded components of our representations are: $F_2^* \otimes F_3^*$, $F_2 \otimes F_1$ and F_1^*. These three can be thought of as three maps

$$F_3^* \otimes \hat{R}_{gen} \overset{d_3^{top}}{\to} F_2 \otimes \hat{R}_{gen} \overset{d_2^{top}}{\to} F_1^* \otimes \hat{R}_{gen} \overset{d_1^{top}}{\to} \hat{R}_{gen}.$$

It is not difficult to see they form a complex over $\hat{R}_{gen}$.

Let us look at Conjecture 21.1 in this case. It means that if we want to see the general point in U_{CM} as a resolution of a perfect ideal, we can do the reverse calculation. We can set the original complex $\mathbf{F}_\bullet$ to be split exact and then, working over polynomial ring in defect variables, we can calculate the differentials d_i^{top} for this complex. If $U_{split} \subset U_{CM}$, then we should get a resolution of a perfect ideal.

Let us do this calculation for this format. We will calculate the top component $v_2^{(3)}$ for the split complex $\mathbf{F}_\bullet$. Let us start with the complex

$$R^{n-3} \overset{d_3}{\to} R^n \overset{d_2}{\to} R^4 \overset{d_1}{\to} R,$$

where

$$d_3 = \begin{pmatrix} 0_{3\times n} \\ I_{n-3} \end{pmatrix}$$

$$d_2 = \begin{pmatrix} I_3 & 0_{3\times n} \\ 0_{1\times 3} & 0_{1\times (n-3)} \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here I_r denotes an $r \times r$ identity matrix and $0_{a\times b}$ is an $a \times b$ zero matrix.

We denote $\{e_1, e_2, e_3, e_4\}$ (resp. $\{f_1 \ldots, f_n\}$, $\{g_1, \ldots, g_{n-3}\}$) the basis in $F_1 \otimes R$ (resp. $F_2 \otimes R$, $F_3 \otimes R$). We will calculate $v_2^{(3)}$. We start with the multiplicative structure. We get

$$e_i e_j = \sum_{s=1}^{n-3} b_{ij;s} f_{s+3}$$

for $1 \le i, j \le 3$,

$$e_i e_4 = -f_i + \sum_{s=1}^{n-3} b_{i4;s} f_{s+3},$$

for $1 \le i \le 3$.

To calculate the top component we need to remember that this is a map which lifts the following cycle

$$\begin{array}{ccccc} \bigwedge^2 F_3 & \to & F_3 \otimes F_2 & \to & S_2F_2 \xrightarrow{S_2(d_2)} S_2F_1 \\ & & \uparrow v_2^{(3)} & \nearrow & S_2(v_1^{(3)}) \\ & & \bigwedge^4 F_1 & & \end{array}.$$

We have

$$S_2(v_1^{(3)})(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = (e_1^{\cdot} e_2)(e_3^{\cdot} e_4) - (e_1^{\cdot} e_3)(e_2^{\cdot} e_4) + (e_2^{\cdot} e_3)(e_1^{\cdot} e_4).$$

So we have

$$v_2^{(3)}(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = (\sum_{s-=1}^{n-3} b_{12;s} g_s) \otimes (-f_3 + \sum_{s=1}^{n-3} b_{34;s} f_s) - (\sum_{s-=1}^{n-3} b_{13;s} g_s) \otimes (-f_2 + \sum_{s=1}^{n-3} b_{24;s} f_s)$$

$$+ (\sum_{s-=1}^{n-3} b_{23;s} g_s) \otimes (-f_1 + \sum_{s=1}^{n-3} b_{14;s} f_s) + \sum_{1 \le s < t \le n-3} c_{st}(g_s \otimes f_{t+3} - g_t \otimes f_{s+3}).$$

Writing this tensor as an $n \times (n-3)$ matrix (which we display in two matrices) we get

$$\begin{pmatrix} -b_{23;1} & -b_{23;2} & \dots \\ +b_{13;1} & +b_{13;2} & \dots \\ -b_{12;1} & -b_{12;2} & \dots \\ c_{1,1} + b_{12;1}b_{34;4} - b_{13;1}b_{24;4} + b_{23;1}b_{14;4} & c_{2,1} + b_{12;2}b_{34;4} - b_{13;2}b_{24;4} + b_{23;2}b_{14;4} & \dots \\ \dots & \dots & \dots \\ c_{1,n-3} + b_{12;1}b_{34;n-3} - b_{13;1}b_{24;n-3} + b_{23;1}b_{14;n-3} & c_{2,n-3} + b_{12;2}b_{34;n-3} - b_{13;2}b_{24;n-3} + b_{23;2}b_{14;n-3} & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots & -b_{23;n-3} \\ \dots & +b_{13;n-3} \\ \dots & -b_{12;n-3} \\ \dots & c_{n-3,1} + b_{12;n-3}b_{34;4} - b_{13;n-3}b_{24;4} + b_{23;n-3}b_{14;4} \\ \dots & \dots \\ \dots & c_{n-3,n-3} + b_{12;n-3}b_{34;n-3} - b_{13;n-3}b_{24;n-3} + b_{23;n-3}b_{14;n-3} \end{pmatrix}$$

Adding to the s-th row W_s the combination $b_{34;s}W_1 - b_{24;s}W_2 + b_{14;s}W_3$ gives us a matrix

$$\begin{pmatrix} -b_{23;1} & -b_{23;2} & \dots & -b_{23;n-3} \\ +b_{13;1} & +b_{13;2} & \dots & +b_{13;n-3} \\ -b_{12;1} & -b_{12;2} & \dots & -b_{12;n-3} \\ c_{1,1} & c_{2,1} & \dots & c_{n-3,1} \\ \dots & \dots & \dots & \dots \\ c_{1,n-3} & c_{2,n-3} & \dots & c_{n-3,n-3} \end{pmatrix}.$$

Notice that the variables $b_{i4;s}$ disappear. They are exactly the variables which, written in the terms of roots of D_n, have label 0 on the vertex n. This is a general phenomenon which happens for other Dynkin formats.

This whole reasoning actually shows that Conjecture21.1 is true for the format $(1, 4, n, n-3)$. Indeed, using linkage it is possible to see (compare [14]) that this gives a generic perfect ideal of this format.

23. Example: format $(1,5,6,2)$.

In this example the grading (E_6,α_3) on algebra $\underline{g}(E_6)$ described in section on gradings.

$$\underline{g}(E_6)=\underline{g}(E_6)_{-2}\oplus\underline{g}(E_6)_{-1}\oplus\underline{g}(E_6)_{0}\oplus\underline{g}(E_6)_{1}\oplus\underline{g}(E_6)_{2},$$

where $\underline{g}(E_6)_0=\underline{sl}(F_3)\times\underline{sl}(F_1)\times\mathbf{C}$ and

$$\underline{g}(E_6)_1=F_3^*\otimes\bigwedge^2 F_1,\underline{g}(E_6)_2=\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1.$$

We get the generic ring from R_a in two steps. First we kill a cycle $r_1^{(3)}$ given by Koszul relations

$$\begin{array}{ccccccccc}0&\to&F_3&\xrightarrow{d_3}&F_2&\xrightarrow{d_2}&F_1&\xrightarrow{d_1}&F_0\\&&&&v_1^{(3)}\nwarrow&&\uparrow\ r_1^{(3)}&&\\&&&&&&\bigwedge^2 F_1&&\end{array}$$

This gives the defect $F_3^*\otimes\bigwedge^2 F_1$. Then we kill the cycle $r_2^{(3)}=S_2(v_1^{(3)})$

$$\begin{array}{ccccccccc}0&\to&\bigwedge^2 F_3&\xrightarrow{d_3}&F_3\otimes F_2&\xrightarrow{d_3}&S_2F_2&\xrightarrow{S_2(d_2)}&S_2F_1\\&&&&v_2^{(3)}\nwarrow&&\uparrow\ r_2^{(3)}&&\\&&&&&&\bigwedge^4 F_1&&\end{array}$$

with defect $\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1$.

At each stage we then divide by annihilators of the ideals $I(d_3)$, $I(d_2)$, $I(d_1)$ and take ideal transforms of the ideals $I(d_3)$, $I(d_2)$. Three critical representations have the form

$$W(d_3)=F_2^*\otimes[F_3\oplus\bigwedge^2 F_1\oplus F_3^*\otimes\bigwedge^4 F_1\oplus\bigwedge^2 F_3^*\otimes S_{2,1,1,1,1}F_1],$$

$$W(d_2)=F_2\otimes[F_1^*\oplus F_3^*\otimes F_1\oplus\bigwedge^2 F_3^*\otimes\bigwedge^3 F_1\oplus S_{2,1}F_3^*\otimes\bigwedge^5 F_1],$$

$$W(d_1)=\mathbf{C}\otimes[F_1\oplus F_3^*\otimes\bigwedge^3 F_1\oplus$$

$$\oplus[\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1\otimes F_1\oplus S_2F_3^*\otimes\bigwedge^5 F_1]\oplus S_{2,1}F_3^*\otimes S_{2,2,1,1,1}F_1\oplus S_{2,2}F_3^*\otimes S_{2,2,2,2,1}F_1]$$

Note that (and this happens for arbitrary format) the lowest weight components (written in brown) are d_3,d_2,a_2 respectively and the next graded components (written in orange) of these three representations are the tensors giving the multiplicative structure on resolution $\mathbf{F}_\bullet$. Moreover (and this is the main point), for any Dynkin format (except for $(1,n,n,1)$ with n odd and $(2,5,5,2)$) three top components (written here in purple) are three differentials in another complex of free modules of the same format over $\hat{R}_{gen}$. We denote this complex by $\mathbf{F}_\bullet^{top}$.

For non-Dynkin formats the critical representations have infinitely many graded components so there are no top components.

Let me finish this section with a remark on the construction of $\hat{R}_{gen}$ from [49].

Remark 23.1. *The ring $\hat{R}_{gen}$ was constructed using the maps p_i giving lifting of some cycles $q_{2,i}$ in the Koszul complex on the maximal minors of d_3 (see Theorem 7.2 in [49]). It is natural to ask where we can see the tensors corresponding to p_i in the ring $\hat{R}_{gen}$. They occur in the subrepresentation $\bigwedge^{r_3} F_2^* \otimes V(\omega_{z_1})$.*

Proof. Indeed, the point is that the graded components $\bigwedge^{r_3} F_3 \otimes \underline{g}_i(T_{p,q,r})$ occur in the i-th graded component of $V(\omega_{z_1})$. The corresponding tensors $\bigwedge^{r_3} F_2^* \otimes \bigwedge^{r_3} F_3 \otimes \underline{g}_i(T_{p,q,r})$ are exactly the maps p_i. □

24. Equivariance Conjecture and its consequences.

The key property of the generic ring (which I expect to be true, but can prove it only for its deformation $\hat{R}_{spec}$, as indicated in the section 18) is.

Conjecture 24.1. *The ring $\hat{R}_{gen}$ has an action of the Lie algebra $\underline{gl}(F_0) \times \underline{gl}(F_2) \times \underline{g}(T_{p,q,r})$. Its decomposition is*

$$\hat{R}_{gen} = \oplus_\mu S_{\phi(\mu)} F_0 \otimes S_{\theta(\mu)} F_2 \otimes V_{\lambda(\sigma(\mu),\tau(\mu),a)}.$$

In this section we draw some consequences of Conjecture 24.1.

Conjecture 24.2. *The ring $\hat{R}_{gen}$ is generated by the following six representations corresponding to the representations $d_1, d_2, d_3, a_1, a_2, a_3$ generating R_a. More precisely, these are*
1) $\alpha = (1), \beta = \gamma = a = b = c = 0$, corresponding to $i = 1$ in Proposition 1.3, 1),
2) $a = 1$, $\alpha = \beta = \gamma = b = c = 0$,
3) $\beta = (1), \alpha = \gamma = a = b = c = 0$, corresponding to $j = 1$ in Proposition 1.3, 3),
4) $b = 1$, $\alpha = \beta = \gamma = a = c = 0$,
5) $\gamma = (1), \alpha = \beta = a = b = c = 0$,corresponding to $k = 1$ in Proposition 1.3, 5),
6) $c = 1$, $\alpha = \beta = \gamma = a = b = 0$.

Proposition 24.3. *Conjecture 24.1 for a given format implies Conjecture 24.2 for the same format.*

Proof. Clearly the representations 1)-6) generate the ring R_a. Thus the multiplication map from a tensor product of two $\underline{gl}(F_0) \times \underline{gl}(F_2)$-isotypic components of $\hat{R}_{gen}$ to a third one is non-zero (and therefore epimorphism) if and only if the same happens to the corresponding $\underline{gl}(F_0) \times \underline{gl}(F_2)$-isotypic components of R_a. But the isotypic components corresponding to representations of types 1)-6) generate R_a which implies our statement. □

Remark 24.4. *In order to show Conjecture 24.2 we need to show that all representations in Proposition 2.3 can be generated by the basic representations of Conjecture 24.2. This should be true because of the following Proposition which shows that representations in Proposition 2.3 corresponding to higher i, j, k just consist of minors in the elements occurring for $i = 1$, $j = 1$, $k = 1$ treated as matrices. I am grateful to Kyu-Hwan Lee for providing a proof of this proposition.*

Proposition 24.5. *Let $V(x_i)$ be a fundamental representation of the Kac-Moody Lie algebra $\underline{g}(T_{p,q,r})$. Then $V(x_i)$ is a factor of $\bigwedge^{p-i}(V(x_{p-1}))$. Similar statement is true for other two arms of $T_{p,q,r}$.*

Proof. The highest weight occurring in $\bigwedge^{p-i}(V(x_{p-1}))$ is ω_{x_i}.

□

The second result is the uniqueness property of $\hat{R}_{gen}$.

Theorem 24.6. *Assume that Conjecture 24.1 is true for the format (f_0, f_1, f_2, f_3).Assume that the format (f_0, f_1, f_2, f_3) is Dynkin or extended Dynkin (i.e. (p, q, r) is $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 5)$). Let $\tilde{R}_{gen}$ be another generic ring for the format (f_0, f_1, f_2, f_3) obtained in a $GL(\mathbb{F}_\bullet)$-equivariant way. Then the natural map $\psi : \tilde{R}_{gen} \to \hat{R}_{gen}$ coming from the universality property of $\tilde{R}_{gen}$ is an epimorphism. This property characterizes $\hat{R}_{gen}$ uniquely, up to na equivariant isomorphism, among all $GL(\mathbb{F}_\bullet)$-equivariant generic rings.*

Proof. We have a homomorphism $\psi : \tilde{R}_{gen} \to \hat{R}_{gen}$, but we also have a homomorphism $\psi' : \hat{R}_{gen} \to \tilde{R}_{gen}$ coming from genericity property.

For the first statement, $\tilde{R}_{gen}$ has to contain tensors p'_i that will cover the same cycles as p_i for $i = 1, 2, \ldots$. The map ψ is equivariant so it has to send p'_i to p_i as these are the only representations we see in $\hat{R}_{gen}$ in the representation containing $\bigwedge^{r_3} d_3$ from R_a as the lowest weight part. Moreover, since $T_{p,q,r}$ is Dynkin or extended Dynkin, we know exactly the components $\underline{g}_i$, they are multiplicity free (see sections 6, 7). So the map from p'_i to p_i has to be an epimorphism. This means the subalgebra R of $\hat{R}_{gen}$ generated by the entries of the p_i's is in the image of the homomorphism ψ. The other elements in $\hat{R}_{gen}$ are in the ideal transform with respect to the ideal $I(d_3)$, so they satisfy the appropriate relations; multiplied by powers of maximal minors of d_3 they are in R. Let W be such representation. The corresponding representation W' has to also occur in $\tilde{R}_{gen}$ as we can apply the homomorphism ψ' to W. Then $\psi(W')$ has to equal to W. So the whole ring $\hat{R}_{gen}$ is in the image of ψ.

The ring $\tilde{R}_{gen}$ is equal to its ideal transform with respect to $I(d_3)$, so it has to contain all the fractions one had to add to $\hat{R}_{gen}$ after adding the matrix entries of maps p_i. Now let R'_{gen} be another generic ring having the uniqueness property of the Theorem. Then two maps $\phi : \hat{R}_{gen} \to R'_{gen}$ and $\psi : R'_{gen} \to \hat{R}_{gen}$ have to be equivariant epimorphisms, so they send p_i to p'_i and vice versa. This means both maps are isomorphisms.

□

Finally we look at the consequences of the existence of generic rings $\hat{R}_{gen}$ for the perfect ideals of codimension 3. For Dynkin formats we analyze the open subsets U_{CM} in $Spec(\hat{R}_{gen})$ of points where the dual of the complex $\mathbb{F}^{gen}_\bullet$ is acyclic.

We also have the following result.

Theorem 24.7. *Assume that the format (f_0, f_1, f_2, f_3) is not Dynkin and that Conjecture 24.1 is true for that format. Then there is no $GL(\mathbb{F}_\bullet)$-equivariant Noetherian generic ring $\tilde{R}_{gen}$ for the format (f_0, f_1, f_2, f_3).*

Proof. It is enough to assume that the format is extended Dynkin, as every non-Dynkin graph $T_{p,q,r}$ contains an extended Dynkin one. It is also clear that the existence of an equivariant Noetherian generic ring for a given format would imply such existence for all smaller formats. In this case we can apply Theorem 24.6. □

The last result might seem bad for non-Dynkin formats. However the situation might not be so hopeless for nice families of resolutions. The following statement that could improve the situation was suggested by Craig Huneke.

Conjecture 24.8. *Assume that S is a graded Noetherian ring. Let $(S, \mathbb{G}(t)_\bullet)_{t\in T}$ be a family of minimal free resolutions over S of format (f_0, f_1, f_2, f_3) of bounded regularity over S. Then there exists an ideal I in $\hat{R}_{gen}$ such that $\hat{R}_{gen}/I$ is Noetherian and such that for all structure maps $\phi_t : \hat{R}_{gen} \to S$ such that $\mathbb{G}(t)_\bullet = \mathbb{F}^{gen}_\bullet \otimes_{\hat{R}_{gen}} S$ we have $I \subset Ker(\phi_t)$.*

25. The branching rules and proof of equivariance property for Dynkin formats

Theorem 25.1. *Conjecture 24.1 is true for all Dynkin formats.*

The strategy for proving Conjecture 24.1 for Dynkin formats is to first do it for type D_n formats and then to use branching rules to extend to other Dynkin formats. We do it only for Dynkin formats with $r_1 = 1$ leaving other cases to the reader. Let us discuss the branching rules first.

We will be dealing with several formats at once so we denote the generic ring corresponding to the rank sequence (r_1, r_2, r_3) by $\hat{R}(r_1, r_2, r_3)_{gen}$.

Consider two Dynkin formats, corresponding to rank sequences (r_1, r_2, r_3) and $(r_1, r_2, r_3 + 1)$. They correspond to triples (p, q, r) and $(p, q, r + 1)$. The generic ring $\hat{R}(r_1, r_2, r_3 + 1)_{gen}$ decomposes as a representation of $\underline{g}(T_{p,q,r})$. We will analyze this situation for Dynkin formats cases by case. The goal is to show in each case that if Theorem 25.1 is true for the triple (p, q, r) it is true for the triple $(p, q, r + 1)$. We refer to such situation as $(r_1, r_2, r_3 \oplus 1)$ case.

We start with the general remarks. Consider the generic ring $\hat{R}(r_1, r_2, r_3 + 1)$ and inside of its spectrum the open set $U(1, d_3)$ of points where the ideal of entries of the matrix d_3 is a unit ideal. Then over the open set $U(1, d_3)$ (locally) our generic complex decomposes

$$0 \to F_3' \oplus R \to F_2' \oplus R \to F_1 \to F_0.$$

Therefore the generic ring $\hat{R}(r_1, r_2, r_3+1)$ on that open set becomes a generic ring $\hat{R}(r_1, r_2, r_3)_{gen}$ extended by some variables. Indeed, defect variables involving the extra summand R in F_3 are free variables.

We denote this situation graphically by denoting the extra branching node by $\square$. Consider this situation case by case.

In each case we will just indicate how the extra simple root of the algebra $\underline{g}(T_{p,q,r+1})$ acts on $\hat{R}_{gen}(r_1, r_2, r_3 + 1)$. Since we know by the proof of the main result that the $\underline{sl}(F_2) \times \underline{sl}(F_0)$ isotypic components of $\hat{R}_{gen}(r_1, r_2, r_3 + 1)$ have the structure of $\underline{g}(T_{p,q,r+1})$-module, this is enough to prove that the multiplication in $\hat{R}_{gen}(r_1, r_2, r_3 + 1)$ is $\underline{g}(T_{p,q,r+1})$-equivariant which is what we want to prove.

Example 25.2. *The case $(1, 4, 1 \oplus 1)$. Graphically we deal with the situation*

$$\begin{array}{ccccccc} \circ & - & \circ & - & \circ & - & \circ \\ & & | & & & & \\ & & \bullet & & & & \\ & & | & & & & \\ & & \square & & & & \end{array}$$

This involves the E_6 format $(1,5,6,2)$ and its branching as a representation of $\underline{g}(D_5)$ Lie algebra. The defects are

$$F_3^* \otimes \bigwedge^2 F_1 \oplus \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1.$$

This means the extra variables coming from decomposition $F_3 = F_3' \oplus R$ are $\bigwedge^2 F_1$ and $\bigwedge^4 F_1$, as the defect for $\underline{g}(D_5)$ is just $\bigwedge^2 F_1$. There is one more extra variable in degree zero corresponding a positive root that have label 1 at the $\square$ node and label 0 at the $\bullet$ node. So altogether we have the representation $\bigwedge^{even}(F_1)$. This is a half-spinor representation of $\underline{g}(D_5)$. Thus we see that the Lie algebra $\underline{g}(D_5)$ acts on $\mathcal{O}_{U(1,d_3)}$. This action descends to the action on the ring $\hat{R}(1,4,2)_{gen}$, as the complement of $U(1,d_3)$ has high codimension. But the subalgebra $\underline{gl}(F_3)$ also acts on $\hat{R}(1,4,2)_{gen}$. These two actions together generate the action of $\underline{g}(E_6)$ on $\hat{R}(1,4,2)_{gen}$.

Example 25.3. *The case $(1,5,1\oplus 1)$. Graphically our situation is denoted*

$$\begin{array}{ccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & | & & & & & & \\ & & \bullet & & & & & & \\ & & | & & & & & & \\ & & \square & & & & & & \end{array}$$

This involves the E_7 format $(1,6,7,2)$ and its branching as a representation of $\underline{g}(D_6)$ Lie algebra. The defects are

$$F_3^* \otimes \bigwedge^2 F_1 \oplus \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1 S_{2,1}F_3^* \otimes \bigwedge^6 F_1.$$

This means the extra variables coming from decomposition $F_3 = F_3' \oplus R$ are $\bigwedge^2 F_1$, $\bigwedge^4 F_1$ and $\bigwedge^6 F_1$, as this is the difference between defects for $(1,5,1)$ and $(1,5,2)$. There is one more extra variable in degree zero corresponding a positive root that have label 1 at the $\square$ node and label 0 at the $\bullet$ node. So altogether we have the representation $\bigwedge^{even}(F_1)$ plus one trivial representation (it consists of one of the copies of $\bigwedge^6 F_1$. This is a half-spinor representation of $\underline{g}(D_6)$. Thus we see that the Lie algebra $\underline{g}(D_6)$ acts on $\mathcal{O}_{U(1,d_3)}$. This action descends to the action on the ring $\hat{R}(1,5,2)_{gen}$. But the subalgebra $\underline{gl}(F_3)$ also acts on $\hat{R}(1,5,2)_{gen}$. These two actions together generate the action of $\underline{g}(E_7)$ on $\hat{R}(1,5,2)_{gen}$.

Example 25.4. *The case $(1,6,1\oplus 1)$. Graphically our situation is denoted*

$$\begin{array}{ccccccccccc}
\circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\
 & & | & & & & & & & & \\
 & & \bullet & & & & & & & & \\
 & & | & & & & & & & & \\
 & & \square & & & & & & & &
\end{array}$$

This involves the E_7 format $(1,7,8,2)$ and its branching as a representation of $\underline{g}(D_7)$ Lie algebra. The defects are

$$F_3^*\otimes\bigwedge^2 F_1\oplus\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1\oplus S_{2,1}F_3^*\otimes\bigwedge^6 F_1\oplus S_{2,2}F_3^*\otimes S_{2,1^6}F_1.$$

This means the extra variables coming from decomposition $F_3=F_3'\oplus R$ are $\bigwedge^2 F_1$, $\bigwedge^4 F_1$ and $\bigwedge^6 F_1$, as this is the difference between defects for $(1,6,1)$ and $(1,6,2)$. There is one more extra variable in degree zero corresponding a positive root that have label 1 at the $\square$ node and label 0 at the $\bullet$ node. So altogether we have the representation $\bigwedge^{even}(F_1)$ plus vector representation of D_7 (it consists of one of the copies of $\bigwedge^6 F_1$ and $S_{2,1^6}F_1$).. This is a half-spinor representation of $\underline{g}(D_7)$. Thus we see that the Lie algebra $\underline{g}(D_7)$ acts on $\mathcal{O}_{U(1,d_3)}$. This action descends to the action on the ring $\hat{R}(1,6,2)_{gen}$. But the subalgebra $\underline{gl}(F_3)$ also acts on $\hat{R}(1,6,2)_{gen}$. These two actions together generate the action of $\underline{g}(E_8)$ on $\hat{R}(1,6,2)_{gen}$.

This completes the branching for the formats of type 2. Notice that extra variables were always the positive part of the Lie algebra $\underline{g}(T_{p,q,r})$ in the grading coming from distinguishing the root cooresponding to the node denoted by $\square$.

We proceed to describe the situation for the two Dynkin formats with five generators.

Example 25.5. *The case $(1,4,2\oplus 1)$. graphically our situation is denoted*

$$\begin{array}{ccccccc}
\circ & - & \circ & - & \circ & - & \circ \\
 & & | & & & & \\
 & & \bullet & & & & \\
 & & | & & & & \\
 & & \circ & & & & \\
 & & | & & & & \\
 & & \square & & & &
\end{array}$$

Counting the defect variables, and calculating the grading on the Lie algebra $\underline{g}(E_7)$ corresponding to the node denoted by $\square$ we see that there are 27 extra variables, as we get

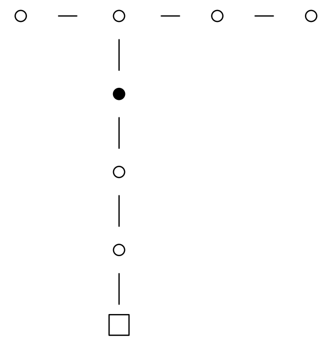
$$\underline{g}(E_7)=V(\omega_6,E_6)\oplus\underline{g}(E_6)\oplus V(\omega_1,E_6).$$

Our defects are:

$$F_3^*\otimes\bigwedge^2 F_1\oplus\bigwedge^2 F_3^*\otimes\bigwedge^4 F_1\oplus\bigwedge^3 F_3^*\otimes S_{2,1^4}F_1.$$

We see 25 extra variables (one copy of $\bigwedge^2 F_1$, two copies of $\bigwedge^4 F_1$ and one copy of $S_{2,1^4}F_1$. There are two more extra variables in degree zero corresponding to two positive roots that have label 1 at the $\square$ node and label 0 at the $\bullet$ node.

Example 25.6. *The case* $(1,4,3\oplus 1)$. *graphically our situation is denoted*



Counting the defect variables, and calculating the grading on the Lie algebra $\underline{g}(E_8)$ *corresponding to the node denoted by* $\square$ *we see that there are* 57 *extra variables, as we get*

$$\underline{g}(E_8) = \mathbb{C}\oplus V(\omega_7, E_7)\oplus \underline{g}(E_6)\oplus V(\omega_7, E_7)\oplus \mathbb{C}.$$

Our defects are:

$$F_3^*\otimes \bigwedge^2 F_1 \oplus \bigwedge^2 F_3^*\otimes \bigwedge^4 F_1 \oplus \bigwedge^3 F_3^*\otimes S_{2,1^4}F_1\oplus \bigwedge^4 F_3^*\otimes S_{2^3,1^2}F_1\oplus S_{2,1^3}F_3^*\otimes S_{2^5}F_1.$$

We see 51 extra variables (one copy of $\bigwedge^2 F_1$*, three copies of* $\bigwedge^4 F_1$*, three copies of* $S_{2,1^4}F_1$*, one copy of* $S_{2^3,1^2}F_1$ *and one copy of* $S_{2^5}F_1$*. There are six more extra variables in degree zero corresponding to two positive roots that have label* 1 *at the* $\square$ *node and label* 0 *at the* $\bullet$ *node.*

The examples above show that in order to show Theorem 25.1 we need to show that for the format $(1,n,n,1)$ the Lie algebra $\underline{g}(D_n)=\underline{sl}(2n)$ acts on the generic ring $\hat{R}(1,n-1,1)_{gen}$.

There are two ways of doing this. One is indicated in the next section and the other is based on another branching which involves the open set $U(1,d_2)$ of complexes where one of the entries of d_2 is a unit. This indicates that our complex $\mathbb{F}_\bullet^{gen}$ partially splits

$$0\to F_3\to F_2'\oplus R\to F_1'\oplus R\to R.$$

We use similar type of reasoning as for the other splitting. Over the set $U(1,d_2)$ the ring $\hat{R}(1,n-1,1)_{gen}$ is a polynomial ring over $\hat{R}(1,n-2,1)_{gen}$ and we count the extra variables in similar way as before. We have $\bigwedge^2 F_1=\bigwedge^2 F_1'\oplus F_1'$, so there are F_1' extra variables. This case of branching is different in that the extra variables occur also in negative degrees, so they form together the vector representation $F_1'\oplus F_1'^*$. So we again get the $\underline{g}(D_{n-1})$-action on $\hat{R}(1,n-1,1)_{gen}$ which, together with obvious $\underline{gl}(F_1)$-action gives us a $\underline{g}(D_n)$-action. This reduces the proof of Theorem 25.1 to the $(1,4,4,1)$ format, but this case was handled in [49], Example 10.5.

We will give a different argument for the D_n formats in Remark 27.2.

26. The symmetry of rings $\hat{R}_{gen}$ in Dynkin cases.

We take a closer look on the structure of $\hat{R}_{gen}$ for a Dynkin format. Start with three critical representations $W(d_3)$ (corresponding to $|\alpha| = 1$, $\beta = \gamma = a = b = c = 0$), $W(d_2)$ (corresponding to $|\beta| = 1$, $\alpha = \gamma = a = b = c = 0$ and $W(d_1)$ (corresponding to $|\gamma| = 1$, $\alpha = \beta = a = b = c = 0$; this has to be replaced by $W(a_2)$ if $r_1 = 1$). These representations have a grading

$$W(d_3) = F_2^* \otimes (\oplus_{i=0}^{top} V(\omega_{z_{r-1}})_i),$$

$$W(d_2) = F_2 \otimes (\oplus_{i=0}^{top} V(\omega_{y_{q-1}})_i),$$

$$W(d_1) = F_0^* \otimes (\oplus_{i=0}^{top} V(\omega_{x_{p-1}})_i).$$

Let us remark that the decompositions of graded components of critical representations to the $\underline{gl}(F_3) \times \underline{gl}(F_1)$ representations are given in [34].

We will employ the following notation. The generators corresponding to graded components of critical representations, when expressed as liftings of cycles in some complexes related to $\mathbb{F}_\bullet^{gen}$ will be denoted as follows.

$$W(d_3) = (v_0^{(3)}, \ldots, v_i^{(3)}, \ldots v_{top}^{(3)}),$$

$$W(d_2) = (v_0^{(2)}, \ldots, v_i^{(2)}, \ldots v_{top}^{(2)}),$$

$$W(d_1) = (v_0^{(1)}, \ldots, v_i^{(1)}, \ldots v_{top}^{(1)}).$$

Of course we have $v_0^{(3)} = d_3$, $v_0^{(2)} = d_2$, $v_0^{(1)} = d_1$ (or $v_0^{(1)} = a_2$ if $r_1 = 1$).

Remark 26.1. *Note that for every rank sequence (r_1, r_2, r_3) the ring $\hat{R}_{gen}$ is tri-graded. Let us assume that $deg(d_3) = \delta_3$, $deg(d_2) = \delta_2$, $deg(a_2) = \delta_1$. Then it is easy to see that we get a grading on $\hat{R}_{gen}$ by setting*

$$deg(v_i^{(3)}) = i\delta_1 - i\delta_2 - (i-1)\delta_3, de(v_i^{(2)}) = i\delta_1 - (i-1)\delta_2 - i\delta_3,$$

$$deg(v_i^{(1)}) = (i+1)\delta_1 - i\delta_2 - i\delta_3.$$

Note that for the format $(1, 6, 8, 3)$ for the pure resolutions with $\delta_1 = 2, \delta_2 = \delta_3 = 1$ we will get all maps $v_i^{(3)}$ and $v_i^{(2)}$ of degree 1 and all maps $v_i^{(1)}$ of degree 2 so we will never et any splitting.

Proposition 26.2. *Let us assume that we have a graded resolution $\mathbb{G}_\bullet$ of length 3 over a polynomial ring S. Denote the basis in G_1 by $\{e_1, \ldots, e_{r_1+r_2}\}$, the basis in G_2 by $\{f_1, \ldots, f_{r_2+r_3}\}$ and the basis in $_3$ by $\{g_1, \ldots, g_{r_3}\}$. Assume that the gradings of terms in the resolution are such that $deg(e_i) = \epsilon_i$, $deg(f_j) = \phi_j$ and $deg(_k) = \psi_k$. Then the degree of every structure map for the resolution $\mathbb{G}_\bullet$ when calculated in homogeneous way is given by calculating its degree as a tensor i.e. taking the degrees of basis vectors in G_1, G_2, G_3 as above, with degree of the dual variable to be negative of the degree of a variable.*

Proof. This follows from the weight structure on $\hat{R}_{gen}$ as all relations preserve the weight decomposition and the degrees given above can be thought of as a linear function on weights. $\square$

For most Dynkin formats three top graded components $v^{(3)}_{top}, v^{(2)}_{top}, v^{(1)}_{top}$ give three tensors which can be put together into a complex

$$\mathbb{F}^{top}_\bullet : F_3^* \to F_2 \to F_1^* \to F_0.$$

This happens for the formats for which three representations $V(\omega_{x_{p-1}})$, $V(\omega_{y_{q-1}})$, $V(\omega_{z_{r-1}})$ are self-dual. There are exceptional formats D_n (n odd) and E_6. In these cases, we have the following. For the D_n format $(1,n,n,1)$, n odd , and for the E_6 format $(2,5,5,2)$ there is no complex $\mathbb{F}^{top}_\bullet$. For the D_n format $(1,n,n,1)$, n even, we have

$$\mathbb{F}^{top}_\bullet : F_3^* \to F_2 \to F_1^* \to F_0.$$

For the D_n format $(n-3,n,4,1)$, we get a complex

$$\mathbb{F}^{top}_\bullet : F_3^* \to F_2 \to F_1 \to F_0.$$

For the D_n format $(1,4,n,n-3)$, we get a complex

$$\mathbb{F}^{top}_\bullet : F_3 \to F_2^* \to F_1^* \to F_0.$$

For the E_6 format $(2,6,5,1)$ we get a complex $\mathbb{F}^{top}_\bullet$ of the form

$$\mathbb{F}^{top}_\bullet : F_3^* \to F_2^* \to F_1^* \to F_0.$$

The first thing to show is that over $\hat{R}_{gen}$ the complex $\mathbb{F}^{top}_\bullet$ is acyclic.

First one has to note that by Peskine-Szpiro Acyclicity Lemma it is enough to do it under assumption that $I(d_3) = R$, i.e. one can invert an $r_3 \times r_3$ minor of d_3. Hence the calculations given below in Remark 27.1, (1) are essential.

Alternate method is to show that the span of highest components in the isotypic components of $\hat{R}_{gen}$ forms a subring isomorphic to R_a. Using the universality of the ring R_a for complexes acyclic in codimension one, it is enough to calculate the split case in a given format to get that there the complex $\mathbb{F}^{top}_\bullet$ is acyclic.

The symmetry of relations in the products of critical representations allows to establish the $GL(\mathbb{F}_\bullet)$-equivariant automorphism θ of a subring of $\hat{R}_{gen}$ generated by the subrepresentations in $W(d_3)$, $W(d_2)$ and $W(a_2)$. The automorphism θ extends to the ideal transform with respect to $I(d_3)$ to give the required automorphism of $\hat{R}_{gen}$.

Still another approach is to to carry out the "reverse geometric construction" to establish the symmetry.

The acyclicity of $\mathbb{F}^{top}_\bullet$ gives a homomorphism of commutative rings $\theta : \hat{R}_{gen} \to \hat{R}_{gen}$ given by the universality property of $\hat{R}_{gen}$, such that

$$\mathbb{F}^{top}_\bullet = (\mathbb{F}^{gen}_\bullet) \otimes_{\hat{R}_{gen}} \hat{R}^{\theta}_{gen}.$$

The map θ is then a $GL(\mathbb{F}_\bullet)$-equivariant automorphism exchanging highest and lowest weight vectors with respect to $\underline{g}(T_{p,q,r})$ and this symmetry implies the $\underline{g}(T_{p,q,r})$ action.

27. Calculation of complexes $\mathbb{F}^{top}_\bullet$.

In this section we calculate the complexes $\mathbb{F}^{top}_\bullet$ in the split case for the cases D_n and the case E_6.

Let us start with some general remarks about the split case. We have three differentials in the canonical form

$$d_3 = \begin{pmatrix} 0 \\ I_{r_3} \end{pmatrix}, d_2 = \begin{pmatrix} I_{r_2} & 0 \\ 0 & 0 \end{pmatrix}, d_1 = \begin{pmatrix} 0 & I_{r_1} \end{pmatrix}.$$

Here I_{r_i} denotes the $r_i \times r_i$ identity matrix.

We denote the basis in F_1 by $\{e_1, \ldots, e_{r_1+r_2}\}$, the basis in F_2 by $\{f_1, \ldots, f_{r_2+r_3}\}$, and the basis in F_3 by $\{g_1, \ldots, g_{r_3}\}$. In the cases $r_1 > 1$ we denote the basis of F_0 by $\{u_1, \ldots, u_{r_1}\}$.

The defect variables in $\mathbb{L}_i$ are denoted by the variables $b(i)$ with appropriate lower and upper indices (lower indices correspond to F_1, upper indices correspond to F_3).

Let us do some sample calculations. Assume that $r_1 = 1$. The map $v_1^{(3)}$ is the piece of the multiplicative structure on our complex. We have

$$e_i^{\cdot}e_j = \sum_{s=1}^{r_3} b(1)^s_{i,j} f_{s+r_3}$$

for $1 \leq i < j \leq r_1$ and

$$e_i^{\cdot}e_{r_1+1} = -f_i + \sum_{s=1}^{r_3} b(1)^s_{i,r_1+1} f_{s+r_3}.$$

Let us calculate the map $v_2^{(3)}$. The second set of defect variables will be denoted $b(2)^{s,t}_{i,j,k,l}$. We have

$$v_2^{(3)}(e_i \wedge e_j \wedge e_k \wedge e_l) = G_{i,j} \otimes (e_k^{\cdot}e_l) - G_{i,k} \otimes (e_j^{\cdot}e_l) + G_{j,k} \otimes (e_i^{\cdot}e_l) +$$
$$+ \sum_{1 \leq s < t \leq r_3} b(2)^{s,t}_{i,j,k,l} d(g_s \wedge g_t)$$

where $1 \leq i < j < k < l \leq r_1 + 1$ and

$$G_{i,j} = \sum_{s=1}^{r_3} b(1)^s_{i,j} g_s.$$

Remark 27.1. *Note that if the split case is calculated, it will be easy to calculate the higher structures for other interesting resolutions, as we will know exactly which cycles lift to give higher structure theorems. This should be done in the following cases*
1) The universal complex of length 2 of format $(f_0, f_1, f_2 - f_3)$ plus the split factor $R^{f_3} \to R^{f_3}$.
2) The universal complex of format $(f_0, f_1, f_2 - 1, f_3 - 1)$ plus the split factor $R \to R$.
3) The universal complex of format $(f_0, f_1 - 1, f_2 - 1, f_3)$ plus the split factor $R \to R$.

We proceed to the Dynkin formats case by case.

27.1. **Format** $(1, n, n, 1)$**.** Now let us calculate the format $(1, n, n, 1)$.

The Lie algebra we deal with is $\underline{g}(D_n) = \underline{s}o(F_1^* \oplus F_1)$.

The critical representations are

$$W(d_3) = F_2^* \otimes [\oplus_{k=0}^{\frac{n}{2}} S_{1-k} F_3 \otimes \bigwedge^{2k} F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1],$$

$$W(a_2) = \mathbb{C} \otimes [\sum_{k=0}^{\frac{n}{2}-1} S_{-k}F_3 \otimes \bigwedge^{2k+1} F_1].$$

Let us make one remark on this format $(1,n,n,1)$ before going to the split format. The Bruns cycles are not defined uniquely by their factorization given above. In order to get them defined uniquely, we need to do the liftings

$$\begin{array}{ccccc} S_{k-1}F_3 \otimes F_2 & \to & S_{k-2}F_3 \otimes F_2 \otimes F_2 & \to S_{k-3}F_3 \otimes F_2 \otimes S_2F_2 \\ & v_k^{(3)} \nwarrow & q_k^{(3)} \uparrow & \\ & & \bigwedge^{2k} F_1 & \end{array}$$

The upper row (for $k \geq 2$) is the beginning part of the Schur complex $S_{(2,1^{k-2})}(F_3 \to F_2)$ (we think of F_3 to be in odd degree, F_2-in even degree). The cycle q_k is the k-th graded component of the relation in degree $(2,0)$ described in the next section. Notice that the lifting is unique because the k-th term of the Schur complex $S_{(2,1^{k-2})}(F_3 \to F_2)$ is $S_{k-1,1}F_3 = 0$.

Similarly the maps $v_k^{(1)}$ $(k \geq 1)$ satisfy the relations

$$\begin{array}{ccccc} S_{k-1}F_3 & \to & S_{k-2}F_3 \otimes F_2 & \to S_{k-3}F_3 \otimes \bigwedge^2 F_2 \\ & v_k^{(1)} \nwarrow & q_k^{(1)} \uparrow & \\ & & \bigwedge^{2k+1} F_1 & \end{array}$$

Here the top row is the beginning part of the $(k-1)$-st exterior power $\bigwedge^{k-1}(F_3 \to F_2)$. The cycle $q_k^{(1)}$ is the $(k-1)$-st graded component of the relations involving $W(d_3)$ and $W(d_1)$.

Let us continue to the analysis of the split format.

The split form of the complex is

$$d_3 = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}, d_2 = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}, d_1 = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

The defect variables are $b_{i,j}$ $(1 \leq i < j \leq n)$. The calculation of the generators amounts to calculating Bruns cycles for this complex. Lifting Bruns cycles gives:

$$v_k^{(3)}(e_1 \wedge \ldots \wedge e_{2k}) = Pf(1,2,\ldots,2k;B)g^{k-1} \otimes f_n$$

$$v_k^{(3)}(e_1 \wedge \ldots \wedge e_{2k-1} \wedge e_n) = \sum_{i=1}^{2k}(-1)^{i+1}Pf(1,2,\ldots,\hat{i},\ldots,2k-1;B)g^{k-1} \otimes (-f_i) +$$
$$+Pf(1,2,\ldots,2k-1,n;B)g^{k-1} \otimes f_n$$

$$v_k^{(1)}(e_1 \wedge \ldots \wedge e_{2k+1}) = 0$$

$$v_k^{(1)}(e_1 \wedge \ldots \wedge e_{2k} \wedge e_n) = Pf(1,2,\ldots,2k;B)g^k.$$
$$+Pf(1,2,\ldots,2k-1,n;B)g^{k-1} \otimes f_n$$

In the case of even n we get a complex $\mathbb{F}_\bullet^{top}$ which has (after some row and column operations which eliminate the variables $b_{i,n}$) has matrices

$$\partial_3 = \begin{pmatrix} Pf((n \hat{-} 1), \hat{n}; B) \\ Pf((n \hat{-} 2), \hat{n}; B) \\ \dots \\ Pf(\hat{1}, \hat{n}; B) \\ 0 \end{pmatrix},$$

$$\partial_2 = \begin{pmatrix} 0 & b_{1,2} & b_{1,3} & \dots & b_{1,n} & 0 \\ -b_{1,2} & 0 & b_{2,3} & \dots & b_{2,n} & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -b_{1,n} & -b_{2,n} & -b_{3,n} & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\partial_1 = \partial_3^t$$

27.2. **Format** $(1, 4, n, n-3)$**.** In this case the Lie algebra corresponding to $T_{p,q,r}$ is $\underline{g}(D_n) = \underline{so}(F_3^* \oplus \bigwedge^2 F_1 \oplus F_3)$. We have

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \otimes \bigwedge^{even} F_3^* \oplus F_1 \otimes \bigwedge^{odd} F_3^*],$$

$$W(a_2) = \mathbb{C} \otimes [F_1^* \otimes \bigwedge^{odd} F_3^* \oplus F_1 \otimes \bigwedge^{even} F_3^*].$$

The defect variables are $b_{i,j;k}$ and $c_{k,l}$ where $1 \le i < j \le 4$, $1 \le k, l \le n$.

The top component of $W(d_3)$ is easy to calculate here.

Remark 27.2. *After checking all quadratic relations involving Bruns cycles, and the dual relations (in the sense of highest-lowest weight for* $\underline{g}(D_n)$ *for the formats* $(1, n, n, 1)$ *and* $(1, 4, n, n-3)$*) we can conclude* $\underline{g}(D_n)$ *acts on generic rings in these cases. This gives another way to finish the proof of Conjecture 24.1 given in Section 25.*

27.3. **Format** $(1, 5, 6, 2)$**.** Restriction formulas give in this case:

$$W(d_3) = F_2^* \otimes [F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1 \oplus \bigwedge^2 F_3^* \otimes S_{2,1^4} F_1],$$

$$W(d_2) = F_2 \otimes [F_1^* \oplus F_3^* \otimes F_1 \oplus \bigwedge^2 F_3^* \otimes \bigwedge^3 F_1 \oplus S_{2,1} F_3^* \otimes \bigwedge^5 F_1],$$

$$W(a_2) = \mathbb{C} \otimes [F_1 \oplus F_3^* \otimes \bigwedge^3 F_1 \oplus [\bigwedge^2 F_3^* \otimes \bigwedge^4 F_1 \otimes F_1 \oplus S_2 F_3^* \otimes \bigwedge^5 F_1] \oplus S_{2,1} F_3^* \otimes S_{2^2,1^3} F_1 \oplus S_{2,2} F_3^* \otimes S_{2^4,1} F_1].$$

We know that $v_1^{(3)}$, $v_1^{(2)}$, $v_1^{(1)}$ give the components of multiplicative structure. We know the factorization $v_2^{(3)}$ which is a map $\bigwedge^4 F_1 \to F_3 \otimes F_2$ lifting the Pfaffian-like element

$(e_1e_2)(e_3e_4) - (e_1e_3)(e_2e_4) + (e_1e_4)(e_2e_3)$. The cycle $v_2^{(2)}$ is the lifting

$$\begin{array}{ccccccc} 0 & \to & \bigwedge^2 F_3 & \to & F_3 \otimes F_2 & \to & S_2F_2 \\ & & v_2^{(2)} & \nwarrow & \uparrow\ q_2^{(2)} & & \\ & & & & \bigwedge^3 F_1 \otimes F_2 & & \end{array}$$

where $q_2^{(2)}$ is a cycle $-v_1^{(1)} \otimes 1_{F_2} + v_1^{(2)}v_1^{(3)} + v_2^{(3)}d_2$. Here the first term is the lower order term coming from the fact that in this case the generic ring is generated by just $W(d_3)$ and $W(d_2)$.

We also give the factorization $v_2^{(1)}$. It consists of two parts:

$$\begin{array}{ccccccc} 0 & \to & F_3 \otimes F_3 & \to & (F_3 \otimes F_2) \oplus (F_2 \otimes F_3) & \to & F_2 \otimes F_2 \\ & & {v'}_1^{(2)} & \nwarrow & \uparrow\ {q'}_1^{(2)} & & \\ & & & & \bigwedge^5 F_1 & & \end{array}$$

where ${q'}_1^{(2)}$ is the combination of two maps $v_1^{(3)}v_1^{(1)}$.

The second part of the factorization $v_2^{(1)}$ is

$$\begin{array}{ccccccc} 0 & \to & \bigwedge^2 F_3 & \to & F_3 \otimes F_2 & \to & S_2F_2 \\ & & {v''}_2^{(1)} & \nwarrow & \uparrow\ {q''}_1^{(2)} & & \\ & & & & S_{2,1^3}F_1 & & \end{array}$$

where ${q''}_1^{(2)}$ is a difference of factorizations

$$S_{2,1^3}F_1 \to \bigwedge^3 F_1 \otimes \bigwedge^2 F_1 \to F_3 \otimes F_2$$

and

$$S_{2,1^3}F_1 \to \bigwedge^4 F_1 \otimes F_1 \to F_3 \otimes F_2,$$

where the first factorization involves $v_1^{(2)}v_1^{(3)}$ ad the second one involves $v_2^{(3)} \otimes d_1$.

Let us next comment on the factorization $v_3^{(2)}$. It is a map

$$\begin{array}{ccccccc} 0 & \to & S_{2,1}F_3 & \to & F_3 \otimes F_3 \otimes F_2 & \to & F_3 \otimes F_2 \otimes F_2 \\ & & v_3^{(2)} & \nwarrow & \uparrow\ q_3^{(2)} & & \\ & & & & \bigwedge^5 F_1 \otimes F_2 & & \end{array}$$

where the cycle $q_3^{(2)}$ is a combination of $v_1^{(3)}v_2^{(2)}$, $v_2^{(3)}v_1^{(2)}$ and the lower order term ${v'}_2^{(1)} \otimes 1_{F_2}$. The explanation of lower order terms in $\underline{g}(T_{p,q,r})$-equivariant terms is given in the next section.

There are two remaining factorizations: $v_3^{(1)}$ and $v_4^{(1)}$. Notice, that they can be calculated by "double reverse" calculation as a differential and a part of multiplicative structure of the resolution of the Schubert variety we see in U_{split}.

The factorization $v_3^{(1)}$ comes from the diagram

$$\begin{array}{ccccccc} 0 & \to & S_{2,1}F_3 & \to & F_3 \otimes F_3 \otimes F_2 & \to & F_3 \otimes F_2 \otimes F_2 \\ & & v_3^{(1)} & \nwarrow & \uparrow\ q_3^{(1)} & & . \\ & & & & \bigwedge^5 F_1 \otimes \bigwedge^2 F_1 & & \end{array}$$

Here the cycle $q_3^{(1)}$ is a combination of $d_1 v_3^{(3)}$, $v_1^{(1)}v_2^{(3)}$ and $v_2^{(1)}v_1^{(3)}$,
The factorization $v_4^{(1)}$ comes from the diagram

$$\begin{array}{ccccccc} 0 & \to & S_{2,2}F_3 & \to & S_{2,1}F_3\otimes F_2 & \to & S_2F_3\otimes\bigwedge^2 F_2\oplus\bigwedge^2 F_3\otimes S_2F_2 \\ & & & v_4^{(1)}\ \nwarrow & \uparrow\ q_4^{(1)} & & \\ & & & & \bigwedge^5 F_1\otimes\bigwedge^4 F_1 & & \end{array}.$$

Here the cycle $q_4^{(1)}$ is a combination of $v_1^{(1)}v_3^{(3)}$, $v_2^{(1)}v_2^{(3)}$ and $v_3^{(1)}v_1^{(3)}$.

Remark 27.3. *After checking these relations and dual relations (in the sense of highest-lowest weight duality for E_6) the format $(1,5,6,2)$ we can conclude that $\underline{g}(E_6)$ acts on $\hat{R}_{gen}$ in this case.*

28. Generators and relations of the generic rings $\hat{R}_{gen}$ for Dynkin formats

In this section we give the $\underline{g}(T_{p,q,r})$-equivariant forms of the generic rings $\hat{R}_{gen}$ for the Dynkin formats. This includes the explicit description of the quadratic relations between the components $W(d_3)$ and $W(d_2)$. These are not all relations satisfied by generic rings (such relations mirror defining relations of the rings R_a which are not quadratic), but they suffice to describe $Spec(\hat{R}_{gen})$ in $\underline{s}l(F_2)\times\underline{s}l(F_0)\times\underline{g}(T_{p,q,r})$ equivariant way.

28.1. **Type D_n.** Before we pass to these formats we need to recall some formulas involving the fundamental representations $V(\omega_1)$, $V(\omega_{n-1})$, $V(\omega_n)$. They are proved in Adams book [1], but they go back to Elie Cartan lectures [12] from 1937.

$$\bigwedge^2 V(\omega_1)=V(\omega_2)$$

$$S_2V(\omega_1)=V(2\omega_1)\oplus\mathbb{C}$$

$$\bigwedge^2 V(\omega_{n-1})=\oplus_i V(\omega_{n-2-4i})$$

$$S_2V(\omega_{n-1})=V(2\omega_{n-1})\oplus\oplus_i V(\omega_{n-4i})$$

$$\bigwedge^2 V(\omega_n)=\oplus_i V(\omega_{n-2-4i})$$

$$S_2V(\omega_n)=V(2\omega_n)\oplus\oplus_i V(\omega_{n-4i})$$

with the convention that $V(\omega_0)=\mathbb{C}$. We also have

$$V(\omega_1)\otimes V(\omega_{n-1})=V(\omega_1+\omega_{n-1})\oplus V(\omega_n),$$

$$V(\omega_1)\otimes V(\omega_n)=V(\omega_1+\omega_n)\oplus V(\omega_{n-1}),$$

$$V(\omega_{n-1})\otimes V(\omega_n)=V(\omega_{n-1}+\omega_n)\oplus\oplus_{i\ge 1}V(\omega_{n-1-2i})$$

again with the convention that $V(\omega_0)=\mathbb{C}$.

28.1.1. *Format* $(1,n,n,1)$. We deal with the special orthogonal Lie algebra $\underline{so}(F_1 \oplus F_1^*)$. The Dynkin diagram is

$$\begin{array}{ccccccccccccc} n-1 & - & n-2 & - & n-3 & - & \dots & - & 2 & - & 1 \\ & & | & & & & & & & & \\ & & n & & & & & & & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1, b-c+\beta_2, \dots, b-c+\beta_{n-2}, b-c, -a+b-c)} F_2 \otimes$$

$$V\begin{pmatrix} b & \beta_{n-2} & \beta_{n-3}-\beta_{n-2} & \dots & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & & & \end{pmatrix}.$$

Three critical representations are

$$W(d_3) = F_2^* \otimes V(\omega_{n-1}),$$

$$W(d_2) = F_2 \otimes V(\omega_1),$$

$$W(a_2) = \mathbb{C} \otimes V(\omega_n).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,\dots,0,-a)} F_2 \otimes$$

$$V\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ & a & & & & \end{pmatrix}.$$

The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1, b-c+\beta_2, \dots, b-c+\beta_{n-2}, b-c, b-c)} F_2 \otimes$$

$$V\begin{pmatrix} b & \beta_{n-2} & \beta_{n-3}-\beta_{n-2} & \dots & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & 0 & & & & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in

$$F_2^* \otimes V(\omega_{n-1}) \otimes F_2 \otimes V(\omega_1)$$

given by

$$\mathbb{C} \otimes V(\omega_1 + \omega_{n-1}) \oplus S_{(1,0,\dots,0,-1)} F_2 \otimes V(\omega_n).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_{n-1}) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_1)$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_n)$. The second summand says that the tensor from $V(\omega_{n-1}) \otimes V(\omega_1)$ has in the F_2-traceless part the $V(\omega_n)$-symmetry. This condition needs to be investigated further.

28.1.2. *Format* $(1,4,n,n-3)$. We deal with the special orthogonal Lie algebra $\underline{so}(\bigwedge^2 F_1 \oplus F_3 \oplus F_3^*)$. The Dynkin diagram is

$$\begin{array}{ccccc} n & - & n-2 & - & n-1 \\ & & | & & \\ & & n-3 & & \\ & & | & & \\ & & \dots & & \\ & & | & & \\ & & 1 & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c,-a+b-c,-a+b-c-\alpha_{n-4},\dots,-a+b-c-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_2 & \beta_1-\beta_2 \\ & a & \\ & \alpha_{n-4} & \\ & \alpha_{n-3}-\alpha_{n-4} & \\ & \dots & \\ & \alpha_1-\alpha_2 & \end{pmatrix}.$$

Three critical representations are

$$W(d_3) = F_2^* \otimes V(\omega_1),$$
$$W(d_2) = F_2 \otimes V(\omega_{n-1}),$$
$$W(a_2) = \mathbb{C} \otimes V(\omega_n).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,0,-a,-a-\alpha_{n-4},-a-\alpha_{nn-3},\dots,-a-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} 0 & 0 & 0 \\ & a & \\ & \alpha_{n-4} & \\ & \alpha_{n-3}-\alpha_{n-4} & \\ & \dots & \\ & \alpha_1-\alpha_2 & \end{pmatrix}.$$

The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c,0,\dots,0)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_2 & \beta_1-\beta_2 \\ & 0 & \\ & 0 & \\ & 0 & \\ & \dots & \\ & 0 & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in

$$F_2^* \otimes V(\omega_1) \otimes F_2 \otimes V(\omega_{n-1})$$

given by

$$\mathbb{C} \otimes V(\omega_1 + \omega_{n-1}) \oplus S_{(1,0,\dots,0,-1)}F_2 \otimes V(\omega_n).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_1) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_{n-1})$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_n)$. The second summand says that the tensor from $V(\omega_1) \otimes V(\omega_{n-1})$ has in the F_2-traceless part the $V(\omega_n)$-symmetry. This condition needs to be investigated further.

28.2. **Type E_6.**

28.2.1. *Format* $(1,5,6,2)$. We deal with the Lie algebra $\underline{g}(E_6)$.

Before we go further we mention some formulas. They can be calculated using LiE computer algebra program [35].

$$\bigwedge^2 V(\omega_1) = V(\omega_3),$$
$$S_2 V(\omega_1) = V(2\omega_1) \oplus V(\omega_6),$$
$$\bigwedge^2 V(\omega_2) = V(\omega_4) \oplus V(\omega_2),$$
$$S_2 V(\omega_2) = V(2\omega_2) \oplus V(\omega_1 + \omega_6) \oplus \mathbb{C}.$$
$$\bigwedge^2 V(\omega_6) = V(\omega_5),$$
$$S_2 V(\omega_6) = V(2\omega_6) \oplus V(\omega_1).$$

We also have

$$V(\omega_1) \otimes V(\omega_6) = V(\omega_1 + \omega_6) \oplus V(\omega_2) \oplus \mathbb{C},$$
$$V(\omega_1) \otimes V(\omega_2) = V(\omega_1 + \omega_2) \oplus V(\omega_5) \oplus V(\omega_1),$$
$$V(\omega_2) \otimes V(\omega_6) = V(\omega_2 + \omega_6) \oplus V(\omega_3) \oplus V(\omega_6).$$

The Dynkin diagram is

$$\begin{array}{ccccccc} 2 & - & 4 & - & 5 & - & 6 \\ & & | & & & & \\ & & 3 & & & & \\ & & | & & & & \\ & & 1 & & & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1, b-c+\beta_2, b-c+\beta_3, b-c, -a+b-c, -a+b-c-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & \\ & \alpha_1 & & \end{pmatrix}.$$

Three critical representations are

$$W(d_3) = F_2^* \otimes V(\omega_1),$$
$$W(d_2) = F_2 \otimes V(\omega_6),$$
$$W(a_2) = \mathbb{C} \otimes V(\omega_2).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,0,0,-a,-a-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} 0 & 0 & 0 & 0 \\ & a & & \\ & \alpha_1 & & \end{pmatrix}.$$

The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c+\beta_3,b-c,b-c,b-c)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & 0 & & \\ & 0 & & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in

$$F_2^* \otimes V(\omega_1) \otimes F_2 \otimes V(\omega_6)$$

given by

$$\mathbb{C} \otimes V(\omega_1+\omega_6) \oplus S_{(1,0,\dots,0,-1)}F_2 \otimes V(\omega_2).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_1) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_1)$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_2)$. The second summand says that the composition uv is a scalar matrix. This last fact is exceptional for this format.

28.3. **Type E_7.** Many calculations involving representation theory related to root system E_7 were calculated using LiE computer algebra program [35]. We will need the following formulas

$$\bigwedge^2 V(\omega_1) = V(\omega_3) \oplus V(\omega_1),$$
$$S_2V(\omega_1) = V(2\omega_1) \oplus V(\omega_6) \oplus \mathbb{C},$$
$$\bigwedge^2 V(\omega_2) = V(\omega_4) \oplus V(\omega_2+\omega_7) \oplus V(2\omega_1) \oplus V(\omega_6) \oplus \mathbb{C},$$
$$S_2V(\omega_2) = V(2\omega_2) \oplus V(\omega_1+\omega_6) \oplus V(\omega_3) \oplus V(2\omega_7) \oplus V(\omega_1),$$
$$\bigwedge^2 V(\omega_7) = V(\omega_6) \oplus \mathbb{C},$$
$$S_2V(\omega_7) = V(2\omega_7) \oplus V(\omega_1),$$
$$V(\omega_1) \otimes V(\omega_2) = V(\omega_1+\omega_2) \oplus V(\omega_5) \oplus V(\omega_1+\omega_7) \oplus V(\omega_2) \oplus V(\omega_7),$$

$$V(\omega_1)\otimes V(\omega_7)=V(\omega_1+\omega_7)\oplus V(\omega_2)\oplus V(\omega_7),$$
$$V(\omega_2)\otimes V(\omega_7)=V(\omega_2+\omega_7)\oplus V(\omega_3)\oplus V(\omega_6)\oplus V(\omega_1).$$

28.3.1. *Format* $(1,5,7,3)$. We deal with the Lie algebra $\underline{g}(E_7)$.

The Dynkin diagram is

$$\begin{array}{ccccccc} 2 & - & 4 & - & 3 & - & 1 \\ & & | & & & & \\ & & 5 & & & & \\ & & | & & & & \\ & & 6 & & & & \\ & & | & & & & \\ & & 7 & & & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen}=\bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c+\beta_3,b-c,-a+b-c,-a+b-c-\alpha_2,-a+b-c-\alpha_1)}F_2\otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & \\ & \alpha_2 & & \\ & \alpha_1-\alpha_2 & & \end{pmatrix}.$$

Three critical representations are

$$W(d_3)=F_2^*\otimes V(\omega_7),$$
$$W(d_2)=F_2\otimes V(\omega_1),$$
$$W(a_2)=\mathbb{C}\otimes V(\omega_2).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,0,0,-a,-a-\alpha_2,-a-\alpha_1)}F_2\otimes$$
$$V\begin{pmatrix} 0 & 0 & 0 & 0 \\ & a & & \\ & \alpha_2 & & \\ & \alpha_1-\alpha_2 & & \end{pmatrix}.$$

The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c+\beta_3,b-c,b-c,b-c,b-c)}F_2\otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in

$$F_2^*\otimes V(\omega_7)\otimes F_2\otimes V(\omega_1)$$

given by

$$\mathbb{C} \otimes (V(\omega_1 + \omega_7) \oplus V(\omega_7)) \oplus S_{(1,0,\dots,0,-1)} F_2 \otimes (V(\omega_7) \oplus V(\omega_2)).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_7) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_1)$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_2)$. The relation coming from the second summand should be investigated.

28.3.2. *Format* $(1, 6, 7, 2)$. We deal with the Lie algebra $\underline{g}(E_7)$.

The Dynkin diagram is

$$\begin{array}{ccccccccc} 2 & - & 4 & - & 5 & - & 6 & - & 7 \\ & & | & & & & & & \\ & & 3 & & & & & & \\ & & | & & & & & & \\ & & 1 & & & & & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1, b-c+\beta_2, b-c+\beta_3, b-c+\beta_4, b-c, -a+b-c, -a+b-c-\alpha_1)} F_2 \otimes$$

$$V\begin{pmatrix} b & \beta_4 & \beta_3-\beta_4 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & & \\ & \alpha_1 & & & \end{pmatrix}.$$

Three critical representations are

$$W(d_3) = F_2^* \otimes V(\omega_1),$$

$$W(d_2) = F_2 \otimes V(\omega_7),$$

$$W(a_2) = \mathbb{C} \otimes V(\omega_2).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2, 0)$, $(1, 1)$ and $(0, 2)$.

The relations in degree $(2, 0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,0,0,0,-a,-a-\alpha_1)} F_2 \otimes$$

$$V\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & a & & & \\ & \alpha_1 & & & \end{pmatrix}.$$

The relations in degree $(0, 2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1, b-c+\beta_2, b-c+\beta_3, b-c+\beta_4, b-c, b-c, b-c)} F_2 \otimes$$

$$V\begin{pmatrix} b & \beta_4 & \beta_3-\beta_4 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & 0 & & & \\ & 0 & & & \end{pmatrix}.$$

The relations in degree $(1, 1)$ are a subspace in

$$F_2^* \otimes V(\omega_1) \otimes F_2 \otimes V(\omega_7)$$

given by

$$\mathbb{C}\otimes (V(\omega_1+\omega_7)\oplus V(\omega_7))\oplus S_{(1,0,\ldots,0,-1)}F_2\otimes (V(\omega_7)\oplus V(\omega_2)).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_7)\xrightarrow{u}F_2\xrightarrow{v}V(\omega_1)$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_2)$. The relation in the second summand should be further investigated.

28.4. **Type E_8.** Many calculations involving representation theory related to root system E_8 were calculated using LiE computer algebra program [35]. We state the formulas that could be useful

$$\bigwedge^2 V(\omega_1)=V(\omega_3)\oplus V(\omega_1+\omega_8)\oplus V(\omega_7)\oplus V(\omega_8),$$

$$S_2V(\omega_1)=V(2\omega_1)\oplus V(\omega_6)\oplus V(\omega_2)\oplus V(2\omega_8)\oplus V(\omega_1)\oplus\mathbb{C},$$

$$\bigwedge^2 V(\omega_2)=V(\omega_4)\oplus V(\omega_2+\omega_7)\oplus V(2\omega_1+\omega_8)\oplus V(\omega_1+\omega_2)\oplus$$

$$\oplus V(\omega_6+\omega_8)\oplus 2^*V(\omega_1+\omega_7)\oplus V(\omega_2+\omega_8)\oplus 2^*V(\omega_3)\oplus$$

$$\oplus V(3\omega_8)\oplus V(\omega_7+\omega_8)\oplus 2^*V(\omega_1+\omega_8)\oplus 2^*V(\omega_7)\oplus V(\omega_8),$$

$$S_2V(\omega_2)=V(2\omega_2)\oplus V(\omega_1+\omega_6)\oplus V(\omega_3+\omega_8)\oplus V(\omega_1+\omega_2)\oplus$$

$$\oplus V(2\omega_7)\oplus V(\omega_5)\oplus V(\omega_1+2\omega_8)\oplus V(\omega_7)\oplus$$

$$\oplus V(\omega_2+\omega_8)\oplus 2^*V(2\omega_1)\oplus V(\omega_7+\omega_8)\oplus 2^*V(\omega_6)\oplus)$$

$$\oplus V(\omega_1+\omega_8)\oplus V(\omega_2)\oplus 2^*V(2\omega_8)\oplus V(\omega_1)\oplus\mathbb{C}.$$

$$\bigwedge^2 V(\omega_8)=V(\omega_7)\oplus V(\omega_8),$$

$$S_2V(\omega_8)=V(2\omega_8)\oplus V(\omega_1)\oplus\mathbb{C},$$

$$V(\omega_1)\otimes V(\omega_2)=V(\omega_1+\omega_2)\oplus V(\omega_5)\oplus V(\omega_1+\omega_7)\oplus V(\omega_2+\omega_8)\oplus$$

$$\oplus V(2\omega_1)\oplus V(\omega_3)\oplus V(\omega_7+\omega_8)\oplus V(\omega_6)\oplus 2^*V(\omega_1+\omega_8)\oplus$$

$$V(\omega_2)\oplus V(2\omega_8)\oplus V(\omega_7)\oplus V(\omega_1)\oplus V(\omega_8).$$

$$V(\omega_1)\otimes V(\omega_8)=V(\omega_1+\omega_8)\oplus V(\omega_2)\oplus V(\omega_7)\oplus V(\omega_1)\oplus V(\omega_8),$$

$$V(\omega_2)\otimes V(\omega_8)=V(\omega_2+\omega_8)\oplus V(\omega_3)\oplus V(\omega_6)\oplus V(\omega_1+\omega_8)\oplus$$

$$\oplus V(\omega_2)\oplus V(\omega_7)\oplus V(\omega_1),$$

28.4.1. *Format* $(1,5,8,4)$. We deal with the Lie algebra $\underline{g}(E_8)$.

The Dynkin diagram is

$$\begin{array}{ccccccc} 2 & - & 6 & - & 7 & - & 8 \\ & & | & & & & \\ & & 5 & & & & \\ & & | & & & & \\ & & 4 & & & & \\ & & | & & & & \\ & & 3 & & & & \\ & & | & & & & \\ & & 1 & & & & \end{array}$$

The decomposition of the generic ring is

$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c+\beta_3,b-c,-a+b-c,-a+b-c-\alpha_3,-a+b-c-\alpha_2,-a+b-c-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & \\ & \alpha_3 & & \\ & \alpha_2-\alpha_3 & & \\ & \alpha_1-\alpha_2 & & \end{pmatrix}.$$

Three critical representations are

$$W(d_3) = F_2^* \otimes V(\omega_1),$$
$$W(d_2) = F_2 \otimes V(\omega_8),$$
$$W(a_2) = \mathbb{C} \otimes V(\omega_2).$$

The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition

$$\bigoplus S_{(0,0,0,0,-a,-a-\alpha_3,-a-\alpha_2,-a-\alpha_1)}F_2 \otimes$$
$$V\begin{pmatrix} 0 & 0 & 0 & 0 \\ & a & & \\ & \alpha_3 & & \\ & \alpha_2-\alpha_3 & & \\ & \alpha_1-\alpha_2 & & \end{pmatrix}.$$

The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition

$$\bigoplus S_{(b-c+\beta_1,b-c+\beta_2,b-c+\beta_3,b-c,b-c,b-c,b-c)}F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_3 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & 0 & & \\ & 0 & & \\ & 0 & & \\ & 0 & & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in
$$F_2^* \otimes V(\omega_1) \otimes F_2 \otimes V(\omega_8)$$
given by all representations except
$$S_{1,0^6,-1} F_2 \otimes V(\omega_1+\omega_8) \oplus \mathbb{C} \otimes V(\omega_2)$$
The first summands involving $\mathbb{C}$ give a condition that says (if we interpret two tensors as maps)
$$V(\omega_1) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_8)$$
that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_2)$. The relation coming from the second summand should be investigated.

28.4.2. *Format* $(1,7,8,2)$. We deal with the Lie algebra $\underline{g}(E_8)$.

The Dynkin diagram is
$$\begin{array}{ccccccccccc} 2 & - & 6 & - & 5 & - & 4 & - & 3 & - & 1 \\ & & | & & & & & & & & \\ & & 7 & & & & & & & & \\ & & | & & & & & & & & \\ & & 8 & & & & & & & & \end{array}$$
The decomposition of the generic ring is
$$\hat{R}_{gen} = \bigoplus S_{(b-c+\beta_1, b-c+\beta_2, b-c+\beta_3, b-c+\beta_4, b-c+\beta_5, b-c, -a+b-c, -a+b-c-\alpha_1)} F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_5 & \beta_4-\beta_5 & \beta_3-\beta_4 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & a & & & & \\ & \alpha_1 & & & & \end{pmatrix}.$$
Three critical representations are
$$W(d_3) = F_2^* \otimes V(\omega_8),$$
$$W(d_2) = F_2 \otimes V(\omega_1),$$
$$W(a_2) = \mathbb{C} \otimes V(\omega_2).$$
The generic ring $\hat{R}_{gen}$ is generated by $W(d_3)$ and $W(d_2)$. We are interested in the relations in degrees $(2,0)$, $(1,1)$ and $(0,2)$.

The relations in degree $(2,0)$ define the subring generated by $W(d_3)$. It has decomposition
$$\bigoplus S_{(0,0,0,0,0,-a,-a-\alpha_1)} F_2 \otimes$$
$$V\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & a & & & & \\ & \alpha_1 & & & & \end{pmatrix}.$$
The relations in degree $(0,2)$ define the subring generated by $W(d_2)$. It has decomposition
$$\bigoplus S_{(b-c+\beta_1, b-c+\beta_2, b-c+\beta_3, b-c+\beta_4, b-c+\beta_5, b-c, b-c, b-c)} F_2 \otimes$$
$$V\begin{pmatrix} b & \beta_5 & \beta_4-\beta_5 & \beta_3-\beta_4 & \beta_2-\beta_3 & \beta_1-\beta_2 \\ & 0 & & & & \\ & 0 & & & & \end{pmatrix}.$$

The relations in degree $(1,1)$ are a subspace in

$$F_2^* \otimes V(\omega_8) \otimes F_2 \otimes V(\omega_1)$$

given by all represetations except

$$\mathbb{C} \otimes V(\omega_2) \oplus S_{(1,0^6,-1)} F_2 \otimes V(\omega_1 + \omega_8).$$

The first summand gives a condition that says (if we interpret two tensors as maps)

$$V(\omega_8) \xrightarrow{u} F_2 \xrightarrow{v} V(\omega_1)$$

that the composition vu defines a tensor which is in the highest weight orbit in $V(\omega_2)$. The relation in the second summand should be further investigated.

29. Perfect ideals of codimension 3 and Schubert varieties.

The examples we saw for the D_n formats show that indeed the generic point of U_{split} the complex $\mathbf{F}_\bullet^{top}$ gives a nice resolution of perfect ideal. It turns out these ideals are defining ideals of certain Schubert varieties. This pattern (first obtained in [41]) can be described as follows.

For a Dynkin diagram of type $T_{p,q,r}$ we can associate to each node x_i a homogeneous space $G(T_{p,q,r})/P_{x_i}$ where $G(T_{p,q,r})$ is a simply connected reductive complex group associated to $T_{p,q,r}$ and P_{x_i} is a parabolic subgroup whose Lie algebra is the parabolic subalgebra

$$\underline{g}(T_{p,q,r})_{\geq 0} = \oplus_{j \geq 0} \underline{g}_j(T_{p,q,r})$$

where the grading is the one associated to α_i. The easiest example are Grassmannians.

Example 29.1. *Let $T_{p,q,r} = A_n$, let us choose the node x_r. x_r. The space $G(A_n)/P_{x_r}$ is the Grassmannian $Grass(r, \mathbb{C}^{n+1})$.*

Let us look more closely at Schubert varieties in $G(T_{p,q,r})/P_{x_i}$. They correspond to elements of the set of cosets $W(T_{p,q,r})/W_{P_{x_i}}$. Here $W_{P_{x_i}}$ is a subgroup of $W(T_{p,q,r})$ generated by reflections different from s_{x_i}. One can look at it in a different way: the set $W(T_{p,q,r})/W_{P_{x_i}}$ is in bijection with the $W(T_{p,q,r})$-orbit of the fundamental weight ω_i.

Example 29.2. *Let us consider the Grassmannian $Grass(4, \mathbb{C}^6)$. This situation corresponds to the diagram*

$$\circ \; - \; \circ \; - \; \circ \; - \; \bullet \; - \; \circ$$

The $W(A_5)$-orbit of the fundamental weight $(0,0,0,1,0)$ consists of 15 elements. The traditional way of parametrizing Schubert subvarieties in the Grassmannian $Grass(4, \mathbb{C}^6)$ is by subsets of cardinality 4 in $[1,6]$. We list below the bijection between two subsets. Notice that the Bruhat order is generated in both cases by the action of the reflections s_i. So in the i-th row we get Schubert subvarieties of codimension $i-1$ in $Grass(4, \mathbb{C}^6)$.

$$(0,0,0,1,0) \leftrightarrow \{1,2,3,4\}$$
$$(0,0,1,-1,1) \leftrightarrow \{1,2,3,5\}$$
$$(0,1,-1,0,1) \leftrightarrow \{1,2,4,5\}, (0,0,1,0,-1) \leftrightarrow \{1,2,3,6\}$$
$$(1,-1,0,0,1) \leftrightarrow \{1,3,4,5\}, (0,1,-1,1,-1) \leftrightarrow \{1,2,4,6\}$$
$$(-1,0,0,0,1) \leftrightarrow \{2,3,4,5\}, (1,-1,0,1,-1) \leftrightarrow \{1,3,4,6\}, (0,1,0,-1,0) \leftrightarrow \{1,2,5,6\}$$

$$(-1,0,0,1,-1) \leftrightarrow \{2,3,4,6\}, (1,-1,1,-1,0) \leftrightarrow \{1,3,5,6\}$$
$$(-1,0,1,-1,0) \leftrightarrow \{2,3,5,6\}, (1,0,-1,0,0) \leftrightarrow \{1,4,5,6\}$$
$$(-1,1,-1,0,0) \leftrightarrow \{2,4,5,6\}$$
$$(0,-1,0,0,0) \leftrightarrow \{3,4,5,6\}.$$

It works similarly for the other formats. The bijection from the right hand side to the left hand side assigns to a subset $I \subset [1, n+1]$ the collection $(\lambda_1, \dots, \lambda_n)$ where λ_i is equal to 1 if $i \in I$, $i+1 \notin I$, $\lambda_i = 0$ if both i and $i+1$ are in I or not in I, and $\lambda_i = -1$ if $i \notin I$, $i+1 \in I$.

Let us go through the appropriate examples.

Example 29.3. *Type D_n. Consider the case (D_n, α_n). The Schubert varieties up to codimension 3 are*

$$\begin{array}{ccccccc} 0&0&\dots&0&0&0&0\\ &&&&&1& \end{array}$$

$$\begin{array}{ccccccc} 0&0&\dots&0&0&1&0\\ &&&&&-1& \end{array}$$

$$\begin{array}{ccccccc} 0&0&\dots&0&1&-1&1\\ &&&&&0& \end{array}$$

$$\begin{array}{ccccccc} 0&0&\dots&0&1&0&-1\\ &&&&&0& \end{array}, \quad \begin{array}{ccccccc} 0&0&\dots&1&-1&0&1\\ &&&&&0& \end{array}$$

So in codimension 3 we have two Schubert varieties of codimension 3. We make them explicit below.

Similar pattern occurs for general diagram $T_{2,q,r}$ distinguishing the simple root α_2.

We have

Example 29.4. *Type $T_{2,q,r}$*

$$\begin{array}{cccccccc} 1&0&0&0&\dots&0&0&0\\ &0&&&&&&\\ &0&&&&&&\\ &\dots&&&&&&\\ &0&&&&&& \end{array}$$

$$\begin{array}{cccccccc} -1&1&0&0&\dots&0&0&0\\ &0&&&&&&\\ &0&&&&&&\\ &\dots&&&&&&\\ &0&&&&&& \end{array}$$

$$\begin{array}{cccccccc} 0&-1&1&0&\dots&0&0&0\\ &1&&&&&&\\ &0&&&&&&\\ &\dots&&&&&&\\ &0&&&&&& \end{array}$$

$$\begin{array}{cccccccc} 0 & 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ & 1 & & & & & & \\ & 0 & & & & & & \\ & \dots & & & & & & \\ & 0 & & & & & & \end{array} , \quad \begin{array}{cccccccc} 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ & -1 & & & & & & \\ & 1 & & & & & & \\ & \dots & & & & & & \\ & 0 & & & & & & \end{array}$$

Again we have two Schubert varieties of codimension 3.

Another feature of Schubert varieties is the fact that inside of the graded coordinate ring of $G(T_{p,q,r})/P_{x_i}$, which decomposes as

$$K[G(T_{p,q,r})/P_{x_i}] = \oplus_{d\geq 0} V(d\omega_i, T_{p,q,r}),$$

are given by linear equations in the primitive embedding into the projective space. This embedding is called the generalized Plücker embedding and the coordinates (basis of $V(\omega_i, T_{p,q,r})$) are called the generalized Plücker coordinates. The easiest generalized Plücker coordinates are the so-called extremal ones, i.e. those in the $W(T_{p,q,r})$-orbit of the highest weight ω_i. They are parametrized in the same way as the Schubert varieties themselves.

This leads to the following pattern, coming from the $T_{p,q,r}$ graph. We do the example of graph E_7 but it is the same for other Dynkin formats.

$$\begin{array}{ccccccccc} \otimes & - & \circ & - & {\color{green}\bullet} & - & \circ & - & \circ \\ & & | & & & & & & \\ & & {\color{red}\bullet} & & & & & & \\ & & | & & & & & & \\ & & \circ & & & & & & \end{array}$$

Two bullets denote the nodes corresponding to two linked formats of type E_7

$${\color{green}0 \to R^3 \to R^7 \to R^5 \to R}$$

$${\color{red}0 \to R^2 \to R^7 \to R^6 \to R}$$

The node $\otimes$ shows the maximal parabolic we need to take.

Denote by s_i the simple reflection in the Weyl group corresponding to node i. We have two elements w_1, w_2 of length 3 in W: $w_1 = s_{\color{green}\bullet} s_c s_\otimes$ and $w_2 = s_{\color{red}\bullet} s_c s_\otimes$. We intersect two Schubert varieties X_{w_1}, X_{w_2} corresponding to these Weyl group elements with the the opposite big cell of $G/P_\otimes$ to get two affine varieties Y_{w_1}, Y_{w_2} with ideals I_{w_1}, I_{w_2} in the polynomial ring R which is the coordinate ring of the opposite big cell in $G/P_\otimes$. The R-modules R/I_{w_1} and R/I_{w_2} have resolutions of exactly needed Dynkin formats! Moreover, the ideals I_{w_1}, I_{w_2} are linked in the big opposite cell of this $G/P_\otimes$. Moreover, this link is canonical, as they are linked by the regular sequence given by three Plücker coordinates that are in both ideals, which are the extremal Plücker coordinates corresponding to Weyl group elements $id, s_\otimes, s_c s_\otimes$.

Proposition 29.5. *Type D_n. The opposite big cell can be identified with $n \times n$ skew-symmetric matrices.*

The variety Y_{w_2} is given by submaximal Pfaffians of odd shaped skew-symmetric matrix (if n is even we have to cut out the first row and column).

The variety Y_{w_1} is an almost complete intersection, i.e. its ideal has 4 generators, it is given by an $n \times n$ skew-symmetric matrix and 3 n-vectors (see [CVW1] for precise description). This situation was already described by Lucho Avramov in 1981 and by Anne Brown ([Br]) in 1987.

Proposition 29.6. *Type E_6. We get two twin Schubert varieties with resolutions*

$$0 \to R^2(-7) \to R^6(-5) \to R(-4) \oplus R^4(-3) \to R.$$

They (or, rather, their linear sections) were described in [CJKW]. There is an equivariant form of these ideals. One of them are contained in the representation $\bigwedge^3 F$, dim $F = 6$. The generators come from an SL_6 invariant Δ of degree 4 on $\bigwedge^3 F$) and its partial derivatives with respect to variables x_{456}, x_{356}, x_{256} and x_{156}. The other has generators Δ and its partials with respect to x_{456}, x_{356}, x_{346} and x_{345}. They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{456}, \partial\Delta/\partial x_{356})$.

Proposition 29.7. *Type E_7. The variety Y_{w_1} has a resolution in the graded format*

$$0 \to R^2(-13) \to R^7(-9) \to R(-7) \oplus R^5(-6) \to R.$$

Its linear section is contained in the representation $\bigwedge^3 F$ where dim $F = 7$. The generators come from an SL_7 invariant Δ of degree 7 on $\bigwedge^3 F$ (dim $F = 7$) and its partial derivatives with respect to x_{567}, x_{467}, x_{367}, x_{267}, x_{167}.

The variety Y_{w_2} has a resolution of graded format

$$0 \to R^3(-13) \to R^7(-10) \to R(-7) \oplus R^4(-6) \to R$$

Its linear section is also contained in the representation $\bigwedge^3 F$. The generators come from Δ and its partial derivatives with respect to x_{567}, x_{467}, x_{457}, x_{456}. They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{567}, \partial\Delta/\partial x_{467})$.

Remark 29.8. *These ideals are small enough so these resolutions were handled by Macaulay2, so we can see the differentials!*

Proposition 29.9. *Type E_8 The variety Y_{w_1} has a resolution in the graded format*

$$0 \to R^2(-31) \to R^8(-21) \to R(-16) \oplus R^6(-15) \to R$$

It is contained in the representation $\bigwedge^3 F$ where dim $F = 8$. The generators of its linear section comes (conjecturally as we cannot verify the Schubert variety intersects the appropriate hyperplane in the right dimension) from SL_8 invariant Δ of degree 16 on $\bigwedge^3 F$, dim $F = 8$, and its partial derivatives with respect to variables x_{678}, x_{578}, x_{478}, x_{378}, x_{278}, x_{178}. The variety Y_{w_2} has a resolution of graded format

$$0 \to R^4(-31) \to R^8(-25) \to R(-16) \oplus R^4(-15) \to R$$

Its linear section is also (conjecturally) contained in the representation $\bigwedge^3 F$ and its defining ideal is generated by Δ and its partials with respect to x_{678}, x_{578}, x_{568}, x_{567}. They are linked by a regular sequence $(\Delta, \partial\Delta/\partial x_{678}, \partial\Delta/\partial x_{578})$.

Remark 29.10. *These resolutions are so big they have not been handled by Macaulay2.*

So the main questions are:

Remark 29.11. *(1) one knows that $U_{split} \subset U_{CM}$*

(2) $\mathbf{U_{CM} = U_{split}}$ **Conjecture***: Do we have $U_{CM} = U_{split}$?,*

(3) **Genericity Conjecture***: prove that the split form indeed gives a generic perfect ideal of a given format,*

It was pointed out by C. Polini and B. Ulrich during the August 2020 ICERM workshop on these topics that these conjectures are connected to the question of Peskine-Szpiro (see [24] for the discussion). It says that when R is a local ring and M, N are two finitely generated R-modules such that M has finite projective dimension and $\ell(M \otimes N) < \infty$, then $dim(M) + dim(N) \leq dim(R)$.

Proposition 29.12. *Assume the annswer to the question of Peskine-Szpiro is positive. Then, for any Dynkin format $U_{CM} = U_{split}$ Conjecture implies Genericity Conjecture.*

Proof. Let us fix a Dynkin format for which $U_{CM} = U_{split}$ Conjecture is true. Let $(S, \mathbb{G}_\bullet)$ be a pair where $(S, \underline{m})$ is a local ring and a resolution of our format which resolves S/J for some perfect ideal of codimension 3. We can also assume that $dim\ S = 3$ and the ideal J is $\underline{m}$-primary. We have a ring homomorphism $\phi : \hat{R}_{gen} \to S$. Passing to completion and adding free variables to $\hat{R}_{gen}$ we can assume without loss of generality that ϕ is an epimorphism. Consider the prime ideal $P \in Spec(\hat{R}_{gen})$ which is the image of $\underline{m}$ under the induced maps on the spectra. If $ht(I_P) = 3$ then we are done, as $P \in U_{CM}$. So let us assume that $ht(I_P) = 2$. Take $R = (\hat{R}_{gen})_P$ and let $\tilde{\phi} : R \to S$ be an induced epimorphim. Recall that since $\hat{R}_{gen}$ has rational singularities, R is Cohen-Macaulay. Denote $I := (I_{gen})_P$. Take $M = R/I$, $N = S$. Then $M \otimes N$ has finite length, $dim(M) = dim(R) - 2$, $dim(N) = 3$ so we have a contradiction with the question of Peskine-Szpiro. □

Remark 29.13. *Assume the $U_{CM} = U_{split}$ Conjecture is not true for some Dynkin format, but question of Peskine-Szpiro is true. Then every resolution of perfect ideal of our Dynkin format will be a deformation of the corresponding Schubert variety. So these Schubert varieties still would play an important role in the classification.*

30. Generic lift property

Let us consider a local ring S and the resolution $\mathbb{G}_\bullet$ of a perfect module of codimension 3 over S. Let $\hat{R}_{gen}$ be the generic ring for the format of $\mathbb{G}_\bullet$. We have a universal map

$$\phi : \hat{R}_{gen} \to S$$

such that

$$\mathbb{G}_\bullet = \mathbb{F}^{gen}_\bullet \otimes_{\hat{R}_{gen}} S.$$

Consider the polynomial ring

$$\hat{S} = S[b(i)_I]$$

over S in the variables corresponding to defect variables for our format. Let

$$\hat{\mathbb{G}}_\bullet = \mathbb{G}_\bullet \otimes_S \hat{S}$$

be the complex we get from $\mathbb{G}_\bullet$ by extending the scalars to $\hat{S}$. We have the following crucial result

Theorem 30.1. *There exists a generic lift*

$$\hat{\phi} : \hat{R}_{gen} \to \hat{S}$$

making the diagram

$$\begin{array}{ccc} \hat{R}_{gen} & \xrightarrow{\hat{\phi}} & \hat{S} \\ & \phi \searrow & \downarrow p \\ & & S \end{array}$$

commute for any choice of ϕ. *It is given by calculating the generic forms of all higher structure theorems for the complex* $\mathbb{G}_\bullet$.

Proof. The proof is self-explanatory, we can always lift the structure theorems in a generic way, using defect variables. Different choices lead to renaming the defect variables in a way that is an affine isomorphism. $\square$

31. The deformation given by $\mathbb{F}_\bullet^{toptop}$

We give here the result about the deformation given by the complex $\mathbb{F}_\bullet^{toptop}$. We will work with a fixed Dynkin format $(1, f_1, f_2, f_3)$ of type E_6, E_7, E_8.

Let $S = K[X_1, \ldots, X_m]$ be a polynomial ring over a field K of characteristic 0. Let J be a perfect ideal of codimension 3 over S such that S/J has a finite free resolution

$$\mathbb{G}_\bullet : 0 \to F_3 \to F_2 \to F_1 \to S$$

of the format $(1, f_1, f_2, f_3)$. Let us denote two sets of defect variables for the format $(1, f_1, f_2, f_3)$ by $\{b_I\}$ and $\{b'_I\}$.

We also denote $T' = S[\{b'_I\}]$ and let $\mathbb{H}_\bullet$ be a complex we get by calculating the top complex for the split complex $\mathbb{H}_\bullet$ of format $(1, f_1, f_2, f_3)$, using the set of variables $\{b'_I\}$.

Theorem 31.1. *The following two facts are true.*

1. *the pair* $(S[\{b_I\}, \{b'_I\}], (\mathbb{G}_\bullet^{top})_\bullet^{top})$ *is a deformation of the original pair* $(S, \mathbb{G}_\bullet)$,
2. *the pair* $(S[\{b_I\}, \{b'_I\}], (\mathbb{G}_\bullet^{top})_\bullet^{top})$ *is a deformation of the complex* $\mathbb{F}_\bullet^{top}$ *for the split exact complex* $\mathbb{F}_\bullet$ *of format* $(1, f_1, f_2, f_3)$.
3. *The reverse calculation applied to the split exact complex of the format* $(1, f_1, f_2, f_3)$ *gives the complex* $(\mathbb{F}_\bullet^{top})_\bullet^{top}$ *which is a resolution a licci ideal.*

Proof. Consider the polynomial ring $T = S[\{b_I\}, \{b'_I\}]$ and over ring T consider the complex $\tilde{\mathbb{G}}_\bullet$ with the differentials given by generic lifts of all HST's given by the generic ring $\hat{R}_{gen}$ in that case. Note that the complex $\tilde{\mathbb{G}}_\bullet$ is acyclic by Peskine-Szpiro Acyclicity Lemma, as in order to prove its acyclicity it is enough to do it for localizations T_P where P are prime ideals with $depth_{T_P} P \leq 3$. But for such prime ideals the localization of the ideal J is a unit ideal. So after this localization we deal essentially with the resolution of our Schubert variety which we know is acyclic. Note that the same argument shows that all ideals J, J^{top} and $(J^{top})^{top}$ over S, T and T' have the corresponding cyclic modules with resolutions of format $(1, f_1, f_2, f_3)$.

We have a diagram

$$\begin{array}{ccccc} & & (T, \tilde{\mathbb{G}}_\bullet) & & \\ & \phi \swarrow & & \searrow \psi & \\ (S, \mathbb{G}_\bullet) & & & & (T', \mathbb{F}_\bullet) \end{array}$$

We need to define maps ϕ and ψ and show that they are both complete intersections. The mapr ϕ is just dividing T by a regular sequence (b_I, b'_I) where b_I, b'_I are two sets of defect variables. The resulting ideal is just the original ideal J. The map ψ is dividing by the ideal $(X_i - x_i^{(0)}, b_I - b_i^{(0)}))$ where $(x_i^{(0)}, b_I^{(0)})$ are coordinates of a particular point in $SpecS[\{b_I\}]$ which is not in the zero set of the ideal J^{top}. This proves the first part of the proposition.

The ideal J^{top} involves only variables X_i and b_I, so we can think of it as the ideal in T', But there its resolution is split, so the complex $\mathbb{F} \otimes_T T'$ is the top complex of a split exact complex. Therefore it is the resolution of our Schubert variety, i.e. the corresponding ideal is licci.

□

Remark 31.2. *The hope was to use the above theorem to prove LICCI conjecture. The idea was to use the technique of Huneke-Ulrich* [26], [43] *to prove that arbitrary perfect ideal of codimension* 3 *with free resolution of format* $(1, f_1, f_2, f_3)$ *over a polynomial ring is a generalized localization in the sense of Huneke-Ulrich (see for example Lemma 1.11 in* [43]*). But the second deformation, involving specializing the variables* X_i *is unfortunately not local, so this result does not imply LICCI conjecture.*

32. Examples from algebra and geometry

32.1. **Artin algebras and Macaulay inverse systems.** We work with the polynomial ring $S := K[x, y, z]$. It is a graded ring $S = \oplus_{i\geq 0} S_i$. We will consider the homogeneous ideals I such that S/I is an Artin ring. This means certain power of the irrelevant ideal $\mathfrak{m} = \oplus_{i>0} S_i$ is contained in I. We are interested in minimal grade free resolution of S/I over S. It will have length 3. In particular we would like to know when such resolution has Dynkin format.

We define the Hilbert function $h_I(t)$ to be the polynomial

$$h_I(t) = \sum_{i\geq 0} (dim(S/I)_i) t^i.$$

Macaulay inverse system allows to produce interesting ideals I. We start with the dual variables $x' = \partial/\partial x$, $y' = \partial/\partial y$, $z' = \partial/\partial z$. To any subspace $V \subset T := K[x', y', z']$ we associate its orthogonal complement ideal

$$I(V) = \{f \in K[x, y, z] \mid f(V) = 0\}$$

where f acts on T via differential operators.

The basic example is the case of $dim\ V = 1$. Then the ideal $I(V)$ is Gorenstein and one proves that any homogeneous Gorenstein ideal $I \subset S$ arrives in that way.

We are interested in families of subspaces V (where we fix dimension of V in each degree) such that for general choice of element from V the cyclic module $S/I(V)$ has a resolution of Dynkin format. Alternatively, we can look at the families of ideals I with fixed Hilbert function of S/I and try to decide when such family could have resolutions of Dynkin format.

For this one does something called Sample Calculation which is best explained by example:

Example 32.1. *Let us consider the ideals I such that the Hilbert function of S/I is* $(1, 3, 6, 6, 2)$*. What is the expected resolution of S/I ? It is gotten by polynomial*

$$(1 + 3t + 6t^2 + 6t^3 + 2t^4)(1 - 3t + 3t^2 - t^3) = 1 - 4t^3 - t^4 + 5t^5 - 2t^7$$

which means we expect ideals with 4 cubic and one quartic generator, with 5 relations in degree 5 and 2 second syzygies in degree 7.

The point is trying to verify the LICCI Conjecture in such cases, as they were not covered by the results of section 31.

Here are some examples of Hilbert functions of S/I that were produced by Sema Güntürkün:

$(1, 3, 4, 4)$, type E_8, $(1, 3, 5, 3)$, type E_7, $(1, 3, 6, 4, 2)$, type E_7, $(1, 3, 6, 5, 3)$, type E_7. We would like to understand how to produce concrete examples of such ideals by Macaulay 2, and then how to test how they link. Here is the procedure that should in principle work. Let us look at the example.

Example 32.2. *Let us look again at the ideals I such that S/I has the Hilbert function* $(1, 3, 6, 6, 2)$*. In this case it is easy to produce such examples by taking the subspace V of dimension 2 in S_4 and taking the ideal $I(V)$. The resolution one gets is*

$$0 \to S^2(-7) \to S^6(-5) \to S(-4) \oplus S^4(-3) \to S$$

Now in general we have a regular sequence of three elements of degree 3 in I. For a linked ideal J the resolution of S/J will be

$$0 \to S(-5) \oplus S(-6) \to S^5(-4) \to S^2(-2) \oplus S(-3) \to S.$$

But then, in general, if we can find a regular sequence of degrees $2, 2, 3$ in J then we expect one Koszul relation to be among minimal syzygies. This should link to a smaller resolution, so this ideal should be licci.

Such method in principle works numerically, but we are not sure we can always find required regular sequence in low degrees and that the maximal cancellation of ranks in the resulting mapping cone occurs. In fact there are counterexamples to such claims, even in codimension 3 (comp. [25]). However if this occurs in the present case (i.e. 4 cubics generate the ideal of height ≤ 2), then by linking by a regular sequence of degrees $(3, 3, 4)$ we come back to the same graded format and the computer experiments show we can rectify the problem.

Another interesting aspect is trying to find the irreducible components of schemes $Hilb(S, h(t))$ of varieties of homogeneous ideals I such that the Hilbert function $h_I(t)$ is $h(t)$.

There is another interesting phenomenon regarding finite free resolutions that is interesting in this context.

Example 32.3. *Take $h(t) = 1 + 3t + t^2 + t^3 + t^4 + t^5$. The Sample Calculation gives*

$$(1 + 3t + t^2 + t^3 + t^4 + t^5)(1 - 3t + 3t^2 - t^3) = 1 - 5t^2 + 6t^3 - 2t^4 - t^6 + 2t^7 - t^8.$$

So the expected resolution splits into two parts: the resolution of type E_6, resolving the general ideal J with Hilbert function $1 + 3t + \sum_{i \geq 2} t^i$ (the algebra S/J is not Artinian, it is a point) and Koszul complex in two variables given by killing the element in J in degree 6. This *phenomenon happens for some Hilbert functions and it deserves to be investigated. It could produce resolutions of Dynkin formats resolving non-perfect ideals.*

32.2. **Points in $\mathbf{P}^3$.** The ACM sets of points in $\mathbf{P}^3$ also give the length three resolutions. The set of Hilbert functions one could obtain is contained in the Hilbert functions of homogeneous Artinian factors of $K[X, Y, Z]$, but it contains the Hilbert functions of ACM curves of degree d of genus g in $\mathbf{P}^4$ (see below).

32.3. **Curves in $\mathbf{P}^3$.** Non-Cohen-Macaulay curves in $\mathbf{P}^3$ also could lead to resolutions of length three. Giuffrida and Maggioni [20] studied curves lying on a smooth cubic surface in $\mathbf{P}^3$. Exhibiting such examples by Macaulay 2 could also be interesting.

32.4. **ACM curves in $\mathbf{P}^4$.** Caroline and Laurent Gruson in [22] calculated possible Hilbert functions and resolutions of curves of genus g and degree d, up to $d = 15$. There are several cases of Dynkin types, but they seem always to be LICCI. It would be good to confirm it at least for a general curve in such cases.

The smallest cases are: $d = 9, g = 6$ (type E_6), $d = 11, g = 9$ (type E_7), $d = 12, g = 11$ (type E_7), $d = 13, g = 15$ (type E_6).

33. Open problems

33.1. **Dynkin module formats.** One should extend the connection between the resolutions of length 3 and Schubert varieties for Dynkin formats to module formats, i.e. Dynkin formats with $r_1 > 1$. In such cases we still have two Schubert varieties of codimension 3 in the homogeneous space $G(T_{p,q,r})/P(x_1)$. One expects that the generic resolution of a perfect module of that format will be related to certain module supported in the corresponding Schubert variety. The first case to analyze is the format $(2, 5, 5, 2)$ which corresponds to the homogeneous space $G(E_6)/P_3$.

33.2. **Opposite Schubert varieties for infinite types.** One of the main problems is to determine whether the connection between Schubert varieties and resolutions of length 3 extends to infinite cases, i.e. what role is played by two opposite Schubert varieties of codimension 3 that occur in the homogeneous space $G(T_{p,q,r})/P(x_1)$. Here we have manny open cells so we get a hierarchy of such pairs. One wants to determine the relation of the open set U_{CM} and the resolutions that are specializations of ind-varieties in that hierarchy.

33.3. **Representations of $\underline{g}(T_{p,q,r})$.** It is important to determine the restrictions of the representations $V(\omega_{x_{p-1}})$, $V(\omega_{y_{q-1}})$, $V(\omega_{z_{r-1}})$ to Lie algebra $\underline{gl}(F_2) \times \underline{gl}(F_0)$. For finite type cases it is done in [34]. For infinite cases the corresponding formulas seem to be unknown and difficult. But it should be possible to classify the representations whose highest weights lie in a $W(T_{p,q,r})$-orbit of the fundamental weights $\omega_{x_{p-1}}$, $\omega_{y_{q-1}}$, $\omega_{z_{r-1}}$. These correspond to elements in the $W(T_{p,q,r})$-orbit of these three fundamental representations for which all indices at the nodes different than z_1 are nonnegative. These representations should be extremal in the sense that the highest weights of the other representations occurring in the restrictions of $V(\omega_{x_{p-1}})$, $V(\omega_{y_{q-1}})$, $V(\omega_{z_{r-1}})$ are in the convex hull of these extremal representations. Also since these representations occur with multiplicity one, there should have an interpretation in terms of lifting cycles.

We start with some examples. We use the convention of [34] to denote vertices of $T_{p,q,r}$. One can go through Dynkin formats, using the results of [34] and the calculation of $W(T_{p,q,r})$-orbits of fundamental representations carried out by W. Kraśkiewicz [32]. We just give one example.

Example 33.1. *Format* $(1,5,6,2)$*. We treat it as* E_6 *graded by* α_5*.*

In $V(\omega_6)$ *and* $V(\omega_1)$ *all representations are extremal. In* $V(\omega_2)$ *all representations except* $\bigwedge^2 F_3^* \otimes \bigwedge^5 F_1$ *are extremal.*

There is a general procedure to classify such weights.

Start with one such weight, occurring in degree M components of our critical representation. Then apply s_{z_1} (i.e. the distinguished node). This will make the index at node z_1 positive. Then start applying other reflections, at positive indices, in such a way to use neighboring reflection to z_1. This will finally make the index at z_1 positive (we have several choices of doing this). Then apply s_{z_1} again. Index at z_1 will become negative again. Then apply other reflections at other negative indices to get all of them nonnegative (always possible and unique, in our case it is even easy as we deal with product of two A_m systems). This will make the index at z_1 even more negative. So you get another weight of the type we need.

I believe this procedure will give all weights in degree $M+1$ starting from weights in degree M. What I do not know is in how many ways we can do it, how non-unique it is. And of course I would like to have a program that does it.

33.4. **Graded case and LICCI property.** This is to investigate the link between finite order notion and the LICCI property described in section 20. This is still needed to establish LICCI conjecture in Artinian cases.

33.5. **Characteristic free version.** One should develop the characteristic free version of the whole theory. It seems the connection with Schubert varieties should provide it, as they are characteristic free.

33.6. **Depth increasing from 2 to 3 and Lie algebras.** Let (R, I) ba a pair consisting of the commutative ring R and an ideal of depth 2. Can we find a generic ring homomorphism

$$\phi : R \to \hat{R}(R, I)_{gen}$$

such that:

(1) $depth I \hat{R}(R, I)_{gen} \geq 3$,
(2) For every R-algebra S with structure map $f : R \to S$ such that $depth IS \geq 3$ we have a homomorphism $\psi : \hat{R}(R, I)_{gen} \to S$ such that $f = \psi\phi$.

Can we define defect Lie algebra for the problem of finding a generic ring $\hat{R}(R, I)_{gen}$? The generic ring would be constructed by killing cycles in a Koszul complex of I. Again after adding matrix entries of certain cycle factorization (moving from some ring R_n to R'_{n+1}, with $depth IR_n = 2$) we need to increase depth of IR'_{n+1} to 2 by killing the annihilator of IR'_{n+1} and then take the ideal transform to get the ring R_{n+1} with $depth(IR_{n+1}) = 2$.

The main problem seems to be this. If we start with the ring $R = R_0$ we have the first graded component $\mathbb{L}_1$ and we get the ring R_1 (ideal transform of R_0 with matrix entries added, and relations killed). Then when we construct R_2 and the space $\mathbb{L}_2$, the derivations from $\mathbb{L}_1$ extend to R_2 and we get the bracket $\bigwedge^2 \mathbb{L}_1 \to \mathbb{L}_2$. However the bracket is defined over R_2 and it does not have to be an epimorphism. It would be very interesting to find a criterion when it is. In such case one would have a chance to get a similar theory as for finite free resolutions of length 3.

Let me give two examples of raising depth from 2 to 3 when the process works.

Example 33.2. *Consider n generic variables $\{x_1, \dots, x_n\}$ and $m \times n$ generic matrix $A = (a_{i,j})$. Consider the polynomial ring on variables $\{x_j, a_{i,j}\}$ with relations $AX = 0, \bigwedge^{n-1} A = 0$. Equivariantly let F be a free module of rank n over K and let G be a free module of rank m over K. We think of basis elements of F as x_j's and basis elements of $F^* \otimes G$ as $a_{i,j}$'s. Our ring R is therefore*

$$R = Sym(F \oplus F^* \otimes G)/J$$

where J is generated by representations G in bidegree $(1,1)$ and $\bigwedge^{n-1} F^* \otimes \bigwedge^{n-1} G$.

Note that R is the ring of sections of sheaf of algebras

$$\mathcal{S} = Sym(\mathcal{Q} \oplus \mathcal{R}^* \otimes G)/\mathcal{J}$$

where $\mathcal{J}$ is the sheaf of ideals generated by $\bigwedge^{n-1} \mathcal{R}^* \otimes \bigwedge^{n-1} G$. over the Grassmannian $Grass(n-1, F)$ with tautological sequence

$$0 \to \mathcal{R} \to F \to \mathcal{Q} \to 0.$$

Of course

$$\mathcal{S} = \oplus_{d,(\lambda_1,\dots,\lambda_{n-2})} S_d\mathcal{Q} \otimes S_{\lambda_1,\dots,\lambda_{n-2}}\mathcal{R}^* \otimes S_{\lambda_1,\dots,\lambda_{n-2}}G$$

where we sum over $d \in \mathbb{N}$ and partitions λ.

Let I be an ideal in R generated by $x_1, \dots, x_n$.

It is easy to see that $depth(I) = 2$. We want to increase this depth to 3.

Let U be the open set which is the complement of the zero set of I. Let $j : U \to Spec(R)$ be the embedding. Then

$$\mathcal{O}_U = \oplus_{d,(\lambda_1,\dots,\lambda_{n-2})} S_d\mathcal{Q} \otimes S_{\lambda_1,\dots,\lambda_{n-2}}\mathcal{R}^* \otimes S_{\lambda_1,\dots,\lambda_{n-2}}G$$

where we sum over $d \in \mathbb{Z}$ and partitions λ.

Thus $\mathcal{R}^1 j_* \mathcal{O}_U$ is generated by the weight

$$(-2, 0, -1, \dots, -1)$$

which corresponds to $\bigwedge^{n-2} G$. This means that the first homology of the left-hand part of Koszul complex of I of length 3 is generated by the map

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^n F \otimes R & \to & \bigwedge^{n-1} F \otimes R & \to & \bigwedge^{n-2} F \otimes R & \to & \bigwedge^{n-3} F \otimes R \\ & & & & & & \uparrow \bigwedge^{n-2} A & & \\ & & & & & & \bigwedge^{n-2} G \otimes R & & \end{array}$$

Thus we have $\mathbb{L}_1 = \bigwedge^{n-2} G$.

Next consider the calculation of $\mathcal{R}^1 j_* \mathcal{O}_U$.

$$\mathcal{R}^0 j_* \mathcal{O}_U = \oplus_{d \geq 0, \lambda} S_{d,0,-\lambda_{n-2},\dots,-\lambda_1} F \otimes S_\lambda G,$$
$$\mathcal{R}^1 j_* \mathcal{O}_U = \oplus_{d, \lambda} S_{-1,d+1,-\lambda_{n-2},\dots,-\lambda_1} F \otimes S_\lambda G,$$
$$\mathcal{R}^2 j_* \mathcal{O}_U = \oplus_{d, \lambda} S_{-1,-\lambda_{n-2}-1,d+2,\dots,-\lambda_1} F \otimes S_\lambda G,$$

with the convention such that all weights are nonincreasing. This means that the $SL(F)$-trivial isotypic component of the spectral sequence contains $S_0 G$ in $\mathcal{R}^0 j_* \mathcal{O}_U$, $\bigwedge^{n-2} G$ in $\mathcal{R}^1 j_* \mathcal{O}_U$, and $\bigwedge^{n-3} G$ in $\mathcal{R}^2 j_* \mathcal{O}_U$. Thus over $Sym(\bigwedge^{n-2} G)$ there is no homomorphism from $\mathcal{R}^1 j_* \mathcal{O}_U$ to $\mathcal{R}^1 j_* \mathcal{O}_U$. This means (after dualizing) that vanishing of $\mathcal{R}^1 j_* \mathcal{O}_U$ is equivalent to the map

$$\bigwedge^{n-2} G \otimes \mathbb{L}_k \to \mathbb{L}_{k+1}$$

be a monomorphism. This in turns means that $\mathbb{L}$ has to be a universal Lie algebra on $\bigwedge^{n-2} G$, i.e. $U(\mathbb{L})$ is the tensor algebra on $\bigwedge^{n-2} G$. So the cycle killing process is infinite.

33.7. **Finite free resolutions of length n for $n > 3$.** Here is a conjectural pattern for the possible generalization to the free resolutions of length bigger than 3.

33.7.1. *The graphs $\mathbb{T}_{even}(r_1, \ldots, r_n)$ and $\mathbb{T}_{odd}(r_1, \ldots, r_n)$.* Let us fix the format $(r_1, \ldots, r_n)$. We construct from this format two graphs $\mathbb{T}_{even}(r_1, \ldots, r_n)$ and $\mathbb{T}_{odd}(r_1, \ldots, r_n)$ as follows.

The vertices of the graph $\mathbb{T}_{even}(r_1, \ldots, r_n)$ are $x_{n-2i,1}, \ldots, x_{n-2i,f_{n-2i}}$ for $i = 0, 1, \ldots, [\frac{n-1}{2}]$.

There are two kinds of edges: the edges making the vertices $x_{n-2i,1}, \ldots, x_{n-2i,f_{n-2i}}$ into a graph of type $A_{f_{n-2i}}$ and the edges connecting vertices $x_{n-2i,f_{n-2i}}$ to $x_{n-2i-2,r_{n-2i-2}+1}$.

The vertices of the graph $\mathbb{T}_{odd}(r_1, \ldots, r_n)$ are $x_{n-1-2i,1}, \ldots, x_{n-1-2i,f_{n-1-2i}}$ for $i = 0, 1, \ldots, [\frac{n-2}{2}]$.

There are two kinds of edges: the edges making the vertices $x_{n-1-2i,1}, \ldots, x_{n-1-2i,f_{n-1-2i}}$ into a graph of type $A_{f_{n-1-2i}}$ and the edges connecting vertices $x_{n-1-2i,f_{n-1-2i}}$ to $x_{n-1-2i-2,r_{n-1-2i-2}+1}$.

We choose the set of roots S_{even} consisting of the simple roots corresponding to vertices complementary to $x_{n-2i,f_{n-2i}}$ and the set of roots S_{odd} consisting of roots corresponding to vertices complementary to $x_{n-1-2i,f_{n-1-2i}}$. We will refer to the nodes from S_{even} abd S_{odd} as the white nodes, and the nodes $x_{n-2i,f_{n-2i}}$ of $\mathbb{T}_{even}(r_1, \ldots, r_n)$ and the nodes $x_{n-1-2i,f_{n-1-2i}}$ of $\mathbb{T}_{odd}(r_1, \ldots, r_n)$ as the black nodes.

We think of vertices $x_{n-2i,1}, \ldots, x_{n-2i,f_{n-2i}-1}$ as corresponding to the root system of $SL(F_{n-2i})$, and of vertices $x_{n-1-2i,1}, \ldots, x_{n-1-2i,f_{n-1-2i}-1}$ as corresponding to the root system of $SL(F_{n-1-2i})$.

Our indexing is such that there is no vertices corresponding to root system of $SL(F_0)$. We add those vertices as another connected component to the graph $\mathbb{T}_{even}(r_1, \ldots, r_n)$ or $\mathbb{T}_{odd}(r_1, \ldots, r_n)$, whichever is appropriate. For this root system the node x_{2,f_2} does not exist.

Thus the white nodes of $\mathbb{T}_{even}(r_1, \ldots, r_n)$ give the product of root systems for $SL(F_{n-2i})$ and the white nodes of $\mathbb{T}_{odd}(r_1, \ldots, r_n)$ give the product of root systems for $SL(F_{n-1-2i})$.

There is always an edge connecting the root system of F_j to that of F_{j-2} (through the black node at the end) except for $j = 2$.

Consider the Kac-Moody Lie algebras

$$\underline{g}_{even}(r_1, \ldots, r_n) = \underline{g}(\mathbb{T}_{even}(r_1, \ldots, r_n)),$$

$$\underline{g}_{odd}(r_1, \ldots, r_n) = \underline{g}(\mathbb{T}_{odd}(r_1, \ldots, r_n)).$$

The choice of nodes S_{even} and S_{odd} defines gradings on the Kac-Moody Lie algebras $\underline{g}_{even}(r_1, \ldots, r_n)$ and $\underline{g}_{odd}(r_1, \ldots, r_n)$ respectively. The roots corresponding to black nodes have degree 1, the roots corresponding to white nodes have degree 0.

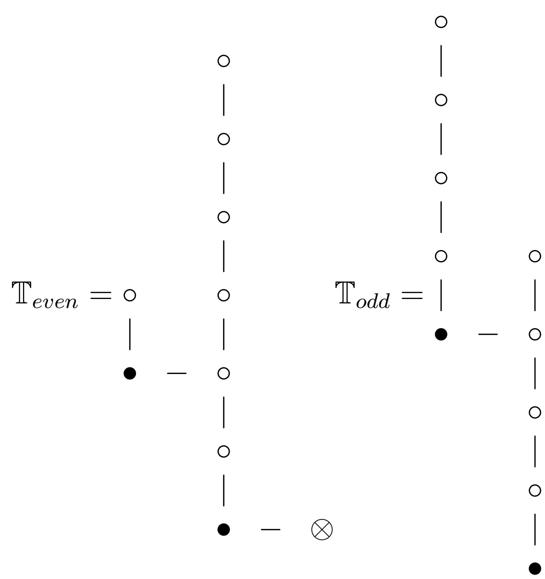
We thus define the Lie algebras $\mathbb{L}_{even}(r_1, \ldots, r_n)$ and $\mathbb{L}_{odd}(r_1, \ldots, r_n)$ to be the positive parts of the Kac-Moody Lie algebras $\underline{g}_{even}(r_1, \ldots, r_n)$ and $\underline{g}_{odd}(r_1, \ldots, r_n)$ respectively.

Example 33.3. *Let $n = 3$. Then the diagram $\mathbb{T}_{even}$ is the graph $T_{p,q,r}$ for $(p, q, r) = (r_1 + 1, r_2 - 1, r_3 + 1)$ considered in [49], and $\mathbb{T}_{odd}$ is just the root system of $SL(F_2) \times SL(F_0)$.*

Example 33.4. *Consider the rank sequence $(1, 4, 3, 2)$, i.e. resolutions*

$$0 \to R^2 \to R^5 \to R^7 \to R^5 \to R.$$

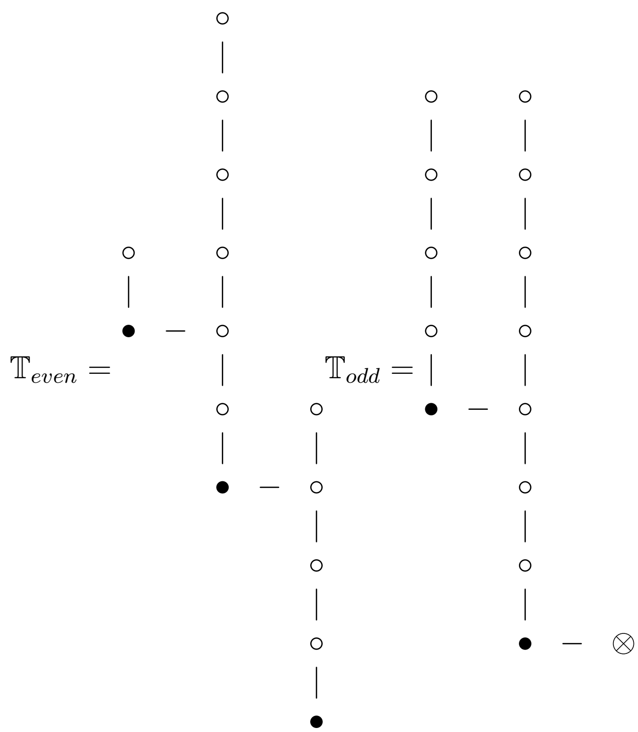
The graphs $\mathbb{T}_{even}(1, 4, 3, 2)$ and $\mathbb{T}_{odd}(1, 4, 3, 2)$ are:



Example 33.5. *Consider the rank sequence $(1, 4, 4, 3, 2)$, i.e. resolutions*

$$0 \to R^2 \to R^5 \to R^7 \to R^8 \to R^5 \to R.$$

The graphs $\mathbb{T}_{even}(1,4,4,3,2)$ *and* $\mathbb{T}_{odd}(1,4,4,3,2)$ *are:*



In both examples $\otimes$ denotes a non-existing node corresponding to root system of $F_0 = R$. I did this to emphasize that each column in $\mathbb{T}_{even}$ and $\mathbb{T}_{odd}$ corresponds to the root system of module F_i for some i. If rank F_0 is bigger than 1, its root system is just a separate connected component. The black nodes indicate extra nodes in the root systems of $\underline{g}(\mathbb{T}_{even})$ and $\underline{g}(\mathbb{T}_{odd})$.

Remark 33.6. *Observe that if* $r_i = 1$ *then there is no node in the Dynkin graph of the root system of* F_i *to which the Dynkin graph of the root system of* F_{i+2} *can be attached. In that case our diagram* $\mathbb{T}$ *is disconnected, as the corresponding map* b_i *does not exist. In practice this will happen only for the last rank in the resolution.*

Notice now that if we look at the Buchsbaum-Eisenbud multiplier ring R_a for our format (see section 15) then for every summand $\lambda \in \Lambda$ in its lattice of weights Λ we have a way of constructing a tensor product of representations $V^{even}(\lambda) \otimes V^{odd}(\lambda)$ of the tensor product $\underline{g}(\mathbb{T}_{even}) \times \underline{g}(\mathbb{T}_{odd})$ by putting on the white nodes the integers coming from the labels in the corresponding root system of F_i, and at the black nodes we put the integers $x^{(i)}$. It seems that there is a good chance that the ring we get

$$\hat{R} = \oplus_{\lambda \in \Lambda} V^{even}(\lambda) \otimes V^{odd}(\lambda)$$

will be a generic ring for the format $(f_0, \ldots, f_n)$. If this is true, it would be the ultimate explanation of the remark at the beginning of section 10 of [10] saying that the first and second structure theorem should determine the free resolution uniquely. It turns out that the whole scheme is really encoded by the First and Second Structure Theorem.

For $n > 3$ we expect that the open set in $Spec\hat{R}_{gen}$ will either be a big open cell, or an open cell in some Schubert variety in the appropriate homogeneous spaces associated to the appropriate Kac-Moody groups G_{even} and G_{odd}. This is because the lifting of cycles of the differential $F_i \to F_{i-1}$ to a map into F_{i+1} is not a free lifting, but a lifting modulo image of F_{i+2}.

34. Appendix: Proof of the existence and properties of the ring $\hat{R}_{gen}$ for $n = 3$

34.1. **Lie algebras and their homology.**

34.1.1. *Generalities.* In this section we work over a fixed field K.

The best references for this material are [4], [28].

For a Lie algebra $\mathbb{L}$ over K we will denote $U(\mathbb{L})$ the enveloping algebra of $\mathbb{L}$, i.e. The factor of a tensor algebra $T(\mathbb{L})$ by the two sided ideal generated by the relations

$$x \otimes y - y \otimes x - [x, y]$$

where $x, y \in \mathbb{L}$.

Let us recall that a graded Lie algebra $\mathbb{L} = \oplus_{i=1}^{\infty} L_i$ is a graded vector space over K with the bracket

$$[,] : \mathbb{L} \otimes \mathbb{L} \to \mathbb{L}$$

such that $[L_i, L_j] \subset L_{i+j}]$. We also require usual Lie algebra properties

(1) $[x, y] = -[y, x]$. for $x \in L_i$, $y \in L_j$,
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for $x \in L_i$, $y \in L_j$, $z \in L_k$,

Note that $\mathbb{L}$ is a Lie algebra in the classical sense, not a graded Lie algebra (with additional signs coming from commuting homogeneous elements).

In a graded situation we will denote $U(\mathbb{L})^*$ the graded dual of the enveloping algebra of $\mathbb{L}$. It is the injective envelope of the trivial $U(\mathbb{L})$-module, so the functor $Hom_{U(\mathbb{L})}(-, U(\mathbb{L})^*)$ defines a duality between finite dimensional $U(\mathbb{L})$-modules.

For the properties of enveloping algebras the reader should consult [17].

Let $\mathbb{L}$ be any Lie algebra. We define cohomology group $H^i(\mathbb{L})$ to be the i-th cohomology group of a complex

$$0 \to \mathbb{L}^* \xrightarrow{\partial} \bigwedge^2 \mathbb{L}^* \xrightarrow{\partial} \bigwedge^3 \mathbb{L}^* \xrightarrow{\partial} \ldots$$

where for $c \in \bigwedge^i \mathbb{L}^*$, $l_1, \ldots, i_{i+1} \in \mathbb{L}$,

$$\partial c(l_1 \wedge l_2 \wedge \ldots \wedge l_{i+1}) = \sum_{j,k} (-1)^{j+k} c([l_j, l_k] \wedge l_1 \wedge \ldots \wedge \hat{l}_j \wedge \ldots \wedge \hat{l}_k \wedge \ldots \wedge l_{i+1}).$$

We are interested mostly in the group $H^2(\mathbb{L})$ which has the following interpretation

$$H^2(\mathbb{L}) = \{\textit{classes of the abelian central extensions of } \mathbb{L}\}.$$

To describe this correspondance let us recall that the abelian central extension of $\mathbb{L}$ comes from an exact sequence

$$0 \to K \to \tilde{\mathbb{L}} \to \mathbb{L} \to 0,$$

where $\tilde{\mathbb{L}}$ is a Lie algebra and K is a one dimensional central ideal in $\tilde{\mathbb{L}}$.

The correspondance associates to the cocycle $c : \bigwedge^2 \mathbb{L} \to K$ the Lie algebra

$$\tilde{\mathbb{L}} = \mathbb{L} \oplus K,$$

with the bracket

$$[(l_1, x), (l_2, y)] = ([l_1, l_2], c(l_1 \wedge l_2)).$$

From this correspondance it follows that there exists a universal central abelian extension $\hat{\mathbb{L}}$ of $\mathbb{L}$. We have

$$\hat{\mathbb{L}} = \mathbb{L} \oplus (H^2\mathbb{L})^*,$$

with the bracket,

$$[(l_1, x), (l_2, y)] = ([l_1, l_2], c'(l_1 \wedge l_2)),$$

where c' is the dual of the embedding of 2-cocycles into $\bigwedge^2 \mathbb{L}^*$.

There is a graded version of this correspondence we will use. Suppose that $\mathbb{L}$ is a graded Lie algebra. Then $H^2(\mathbb{L})$ carries a natural gradation and the m-th graded component of $H^2(\mathbb{L})$ corresponds to the abelian central extensions $\tilde{\mathbb{L}}$ of $\mathbb{L}$ for which the ideal K is contained in the m-th graded component of $\tilde{\mathbb{L}}$. We also et the universal central abelian extension $\hat{\mathbb{L}}_m$ with the ideal in degree m. Additively $\hat{\mathbb{L}}_m$ is $\mathbb{L} \oplus H^2(\mathbb{L})^*_m$, with the bracket being defined as before.

Let us now suppose that

$$\mathbb{L} = L_1 \oplus L_2 \oplus \ldots \oplus L_m.$$

Then we get the universal graded central abelian extension

$$0 \to H^2(\mathbb{L})^*_{m+1} \to \hat{\mathbb{L}}_{m+1} \to \mathbb{L} \to 0$$

with the ideal in degree $m + 1$.

34.1.2. *The quadratic Lie algebras coming from cohomology.* Starting with a two-step graded Lie algebra $\mathbb{L}_2 = L_1 \oplus L_2$, we can now repeat the last construction from the last subsection to get the graded Lie algebra

$$\mathbb{L} = L_1 \oplus L_2 \oplus \ldots,$$

where the spaces L_m are defined by induction as

$$L_{m+1} = H^2(\mathbb{L}_m)_{m+1},$$

where $\mathbb{L}_m = L_1 \oplus L_2 \oplus \ldots \oplus L_m$.

In such case, assume that the original bracket

$$[,] : \bigwedge^2 L_1 \to L_2$$

is an epimorphism. In such case the resulting graded Lie algebra

$$\mathbb{L} = L_1 \oplus L_2 \oplus \ldots$$

can be defined as a graded Lie algebra generated (as a Lie algebra) by L_1 with relations in degree 2 given by $Ker(\bigwedge^2 L_1 \to L_2)$.

Note that we will usually assume that the vector spaces L_i are finite dimensional but often the Lie algebra $\mathbb{L}$ resulting from the last construction is infinite dimensional as it has infinitely many components.

34.2. **Defect Lie algebra and its comparison with $T_{p.q.r}$.** The defect Lie algebra $\mathbb{L}(p,q,r)$ for a given format (p,q,r) had two definitions.

34.3. **Open cells in homogeneous spaces $G(T_{p,q,r})/P_{z_1}$.** Let us work with Dynkin format corresponding to the triple (p,q,r). Let G be a simply connected algebraic group corresponding to the root system $T_{p,q,r}$ and let P_{z_1} be a maximal parabolic subgroup corresponding to the node z_1 in $T_{p,q,r}$. The homogeneous space $G(T_{p,q,r})/P_{z_1}$ is defined to be $Proj(A)$ where A is a graded ring

$$A = \oplus_{d\geq 0} V(d\omega_{z_1}).$$

There is an associated Bruhat graph $Br(p,q,r)$ which is a $W(T_{p,q,r})$-orbit of the fundamental weight ω_{z_1}. This set can be identified with $W(T_{p,q,r})/W(P_{z_1})$. The elements are represented by minimal length representatives in a given coset. The partial order is given by the relation $w < s_i w$ if $\ell(w) < \ell(s_i w)$. The elements of $Br(p,q,r)$ can be thought of as Schubert varieties, with the varieties of codimension s correspond to elements of length s. The partial order is given by inclusion in this language.

The elements w of $Br(p,q,r)$ also corresponds to extremal Plücker coordinates p_w on $G(T_{p,q,r})/P_{z_1}$.

Thus to each element $w \in Br(p,q,r)$ there is the corresponding affine open subset of elments where the corresponding coordinate p_w is nonzero. In particular there is an opposite big open cell

$$Y(p,q,r) := U_{w_0}$$

where w_0 is the element of maximal length in $Br(p,q,r)$. It is an affine space of dimension equal to dimension of $\mathbb{L}(p,q,r)$.

The basis of Lie algebra $\mathbb{L}(p,q,r)$ corresponds to the coordinates in the affine space Y.

Clearly $Y(p,q,r)$ is P_{z_1}-equivariant, so

Proposition 34.1. *The Lie algebra $\mathbb{L}(p,q,r)$ acts by derivations on the coordinate ring $K[Y]$.*

There is also an interesting relation between the homogeneous space $G(T_{p,q,r})/P_{z_1}$ and the homogeneous space $G(T_{p,q,r})/P_{z_{r-1}}$ corresponding to the extremal node on the arm of z_1.

We have an easy to prove fact

Proposition 34.2. *The representation $V(\omega_{z_1})$ is naturally a factor of $\bigwedge^{r-1} V(\omega_{z_{r-1}})$. It occurs in this exterior power with multiplicity* 1.

Proof. The lowest weight in $\bigwedge^{r-1} V(\omega_{z_{r-1}})$ is ω_{z_1}, and it occurs with multiplicity 1. □

This means that the homogeneous space $G(T_{p,q,r})/P_{z_1}$ is contained in the Grassmannian of $r-1$-subspaces in the representation $G(T_{p,q,r})/P_{z_{r-1}}$.

The description of the open cell Y is given by the matrix $M(p,q,r)$. The columns of the matrix $M(p,q,r)$ are the basis of $V(\omega_{z_{r-1}})$. We divide the columns into blocks according to the graded components of the restriction of $V(\omega_{z_1})$ to the root system of $\underline{sl}(F_3) \times \underline{sl}(F_1)$. There are $r-1$ rows of $M(p,q,r)$. Its number is equal to the dimension of the 0-th graded component of $V(\omega_{z_1})$ under this restriction. So

$$M(p,q,r) = \big(X_0 \quad | \quad X_1 \quad | \quad X_2 \quad \ldots \quad | \quad X_s\big)$$

Here X_0 is an $(r-1)\times(r-1)$ identity matrix, and X_i has entries of degree i in defect variables.

Let us illustrate it by the example of E_6.

Example 34.3. *Let us look at $G(E_6)/P_3$. The representation $V(\omega_3)$ is isomorphic to $\bigwedge^2 V(\omega_1)$. The restriction of $V(\omega_1)$ to an $A_1 \times A_4$ root system is*

$$V(\omega_1) = F_3 \oplus \bigwedge^2 F_1 \oplus F_3^* \otimes \bigwedge^4 F_1 \oplus \bigwedge^2 F_3^* \otimes S_{2,1^4} F_1.$$

The defect variables are $\underline{g}_1 = F_3^* \otimes \bigwedge^2 F_1$ and $\underline{g}_2 = \bigwedge^2 F_3^* \otimes \bigwedge^4 F_1$.

The matrix $M(2,3,3)$ is a 2×27 matrix

$$M(2,3,3) = \big(X_0 \quad | \quad X_1 \quad | \quad X_2 \quad | \quad X_3\big)$$

with blocks of sizes $2\times 2, 2\times 10, 2\times 10, 2\times 5$. The rows of $M(2,3,3)$ can be naturally identified with F_3. So each X_i can be treated as a representation of $GL(F_3)\times GL(F_1)$.

The matrix X_0 is a 2×2 identity matrix. To determine the entries of X_1, X_2, X_3 we notice that there has to be a $GL(F_3)\times GL(F_1)$-equivariant action from $\underline{g}_i \otimes X_j \to X_{i+j}$. So X_1 depends linearly on the coordinates from $\underline{g}_1$, and so on. In principle this determines the matrix $M(p,q,r)$ but doing it precisely in all cases in a subject of the ongoing work, and it is the key to understanding the situation.

The matrix $M(2,3,3)$ determines the embedding of open cell in $G(E_6)/P_3$ into the Grassmannian of 2-subspaces in $G(E_6)/P_1$.

34.4. **Lifting cycles procedure and action of the Lie algebra $\mathbb{L}(p,q,r)$.** Above we quoted the result from [47] that the Lie algebra $\mathbb{L}(p,q,r)$ acts on the ring R_∞ defined as a direct limit of the rings R_m. This action was constructed by induction in quite complicated way. It was necessary at the time as the connection with root systems was not anticipated.

Here we give a much simpler proof of this fact.

We work over the fraction field K_0 of the ring $R_0 := R_a$. We tensor all the constructions of lifting cycles with K_0. We obtained the rings $R'_m = R_m \otimes_{R_0} K_0$. But the spectra of the rings R'_m are just the products of the $Spec(K_0)$ with the affine space of defect variables of degrees $\le m$. The ring R'_∞ ahs the spectrum which is the product of spectrum of K_0 with the big open cell in $G(T_{p,q,r})/P_{z_1}$. So by Proposition 34.1 the Lie algebra $\mathbb{L}(p,q,r)$ acts on R'_∞, as the action is K_0 linear since all entries of differential in the basic complex $\mathbb{F}_\bullet$ are constants of derivations in $\mathbb{L}(p,q,r)$.

Proposition 34.4. *The action of $\mathbb{L}(p,q,r)$ on the ring R'_m descends to the action on R_m. The components $\mathbb{L}(p,q,r)_s$ for $s > m$ act trivially on R_m.*

Proof. We proceed by induction on m, the cases $m = 0$ and $m = 1$ being trivial. Assume that we know that the action of $\mathbb{L}(p,q,r)$ on R'_m descends to R_m. We will prove it for R_{m+1}.

Recall that the ring R_{m+1} was constructed from R_m in the following steps:

(1) Add the entries of the factorization p_{m+1} and divide by relations satisfied by all such factoriazations. Denote the resulting ring by $\tilde{R}_{m+1}$.

(2) Take the ideal transform with respect to $I(d_3)I(d_2)$.

To see that applying derivations from $\mathbb{L}(p,q,r)_{m+1}$ to entries of p_{m+1} hase values in R_{m+1}, for a given derivation D, and applying D to the relations defining p_{m+1}, we see that the values are in $\tilde{R}_{m+1}$.

To see that the ideal transform is taken to itself we take an x element in that transform. This element can be written as

$$x = \frac{u_1}{{v_1}^{m_1}} = \ldots = \frac{u_s}{{v_s}^{m_s}}$$

for $u_i \in \tilde{R}_{m+1}$ and v_i being all generators of $I(d_3)I(d_2)$ and $m_1, \ldots, m_s$ being natural numbers. Applying D, using the quotient rule and taking into account that the elements v_i are constants for D, we see that $D(x)$ is in R_{m+1}. This value does not depend on the choice of the expression since it is the value of action of D on R'_{m+1}.

The last statement of the proposition is clear.

□

34.5. **Spectral sequence and its comparison with parabolic BGG.** The next step in considerations above and proving that $\hat{R}_{gen}$ is indeed a generic ring, is the proof that if

$$j_\infty : U_\infty \to Spec(\hat{R}_{gen})$$

is an open embedding, then $R^1 j_{\infty *}(\mathcal{O}_{U_\infty}) = 0$.

The idea was based on spectral sequence from [47]. Again the proof was quite complicated because the connection with root system was not anticipated at the time. Let us give a simpler argument for the convenience of the reader.

We have the realization of the $\mathfrak{g}l(F_2) \times \mathfrak{g}l(F_0)$ isotypic component of $\hat{R}_{gen}$ as an irreducible representation of $\mathfrak{g}(T_{p,q,r})$, given in the section 18.

This gives the (additive) decomposition, given ins section 19

$$\hat{R}_{gen} = \oplus_{\lambda \in \Lambda} S_{\beta(\lambda)} F_2 \otimes S_{\delta(\lambda)} F_0 \otimes V(\alpha(\lambda), \gamma(\lambda)).$$

Expressing all weights in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \alpha, \beta, \gamma$ and considering the corresponding sheaf $\mathcal{F}_\infty$ on the homogeneous space (a product of homogenaous spaces for $GL(F_2) \times GL(F_0)$ and the homogeneous space $G(T_{p,q,r})/P_{z_1}$ of the Kac-Moody group corresponding to $T_{p,q,r}$), we see that the sheaf $\mathcal{F}'_\infty$ corresponding to $\mathcal{O}_{U_\infty}$ has similar decomposition but that $\mathbf{a}$ can take arbitrary integer values.

In order to prove genericity of $\hat{R}_{gen}$ we need to show (Theorem 12.1) that $R^1 j_{\infty *} \mathcal{O}_{U_\infty} = 0$ we need to prove that the first cohomology group $H^1(\mathcal{F}'_\infty)$ of the above sheaf on the homogeneous space is zero. But if there is a contribution to higher cohomology, it has to come from negative value of and this means that in order to get to a dominant weight we will need to apply the reflections on both components of homogeneous space (the one corresponding to F_2 and F_0 and the one corresponding to $T_{p,q,r}$). So such contributions will occur in H^j for $j \geq 2$.

The above decomposition of $\hat{R}_{gen}$ can be obtained directly using the action of positive part of positive part $\mathfrak{g}(T_{p,q,r})$, and using the reasoning from the proof of Lemma 2.4 from [47], as the span of defect variables gives us the coordinates of the open cell in the homogeneous space $G(T_{p,q,r})/P_{z_1}$.

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University of Connecticut, Storrs, USA

Jagiellonian University, Kraków, Poland